Almost Group Envy-free Allocation of Indivisible Goods and Chores

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July 18, 2019

Abstract

We consider a multi-agent resource allocation setting in which an agent's utility may decrease or increase when an item is allocated. We take the group envy-freeness concept that is well-established in the literature and present stronger and relaxed versions that are especially suitable for the allocation of indivisible items. Of particular interest is a concept called group envy-freeness up to one item (GEF1). We then present a clear taxonomy of the fairness concepts. We study which fairness concepts guarantee the existence of a fair allocation under which preference domain. For two natural classes of additive utilities, we design polynomial-time algorithms to compute a GEF1 allocation. We also prove that checking whether a given allocation satisfies GEF1 is coNP-complete when there are either only goods, only chores or both.

1 Introduction

This work studies fair division, a problem studied since the 1950's in which a set of items is to be divided among agents while taking into account the agents' preferences. Researchers in this area aim at defining relevant fairness concepts and develop algorithm satisfying them. Many fairness concepts have been proposed in this perspective and envy-freeness (EF) is now viewed as the gold standard one. An allocation of the items is said to be envy-free, or satisfies no-envy, if for any pair of agents, no agent should prefer what the other got instead of her own share.

Fair division problems are usually classified depending on whether the items to be allocated are divisible or not. Initially, the fair division problem has been described for the division of a unique divisible items. When considering only indivisible items, the problem becomes combinatorial. While an envy-free division of divisible items can always be found in bounded time (Steinhaus, 1949, Aziz and Mackenzie, 2016), it is no longer possible with indivisible items. Imagine having to divide a painting between two persons, obviously the painting can not be cut into two pieces. In any allocation, one person will get the painting and the other nothing which induces envy. To achieve envy-free allocations with indivisible items different solutions can be envisioned. One can introduce one divisible item, usually called money, that will be used to compensate agents. Another possibility is to allow to get rid of some items, the one always inducing envy. In this work we will not consider any of these solutions: we only consider strictly indivisible items all of which will be allocated without any waste. The last solution is then to define relaxations for which we have guarantees even with indivisible items. Envy-freeness up to one item (EF1) has been defined in this respect (Budish, 2011). It requires no envy between any two agents as long as one agent virtually gets rids of one of her goods. We know that an allocation satisfying EF1 can always be computed when items are indivisible (Lipton, Markakis, Mossel, and Saberi, 2004).

However, envy-freeness only applies to pairs of agents and does not provide any fairness guarantees when comparing groups of agents. Berliant, Thomson, and Dunz (1992) extended envy-freeness to groups of agents by introducing *group envy-freeness (GEF)*. It generalizes envy-freeness for equal-sized groups of agents instead of considering only pairs of agents. In addition to generalizing EF, one particularly desirable aspect of group envy-freeness is that it also implies Pareto-optimality, the central concepts for fairness and efficiency respectively. This is particularly interesting since fairness has a well-known tension with efficiency goals (Caragiannis, Kaklamanis, Kanellopoulos, and Kyropoulou, 2012).

Group envy-freeness has been defined for allocation of divisible items. Recently, Conitzer, Freeman, Shah, and Vaughan (2019) generalized group envy-freeness for indivisible items and groups of different size by introducing *group-fairness (GF)*. As for envy-freeness, with indivisible items it is impossible to guarantee group-fairness. They thus proposed two relaxations of group-fairness. These relaxations are similar in spirit to envy-freeness up to one item. They showed that allocations satisfying the relaxations of group-fairness can always be satisfied and can be computed in pseudo-polynomial time.

Most of the literature on fair division assumes that the items to be allocated are "goods" in the sense that agents improve their satisfaction when receiving such items. It is however very common to have to allocate not only positive items but also negative ones. Consider for instance a shared house in which a set of chores have to be performed by the roommates. This problem can be viewed as a instance of the fair division problem where the items are perceived negatively by the agents. In their paper, Conitzer et al. (2019) assumed that the items allocated are "goods" for which agents have positive utility. Therefore the concepts and results do not apply to allocations scenarios in which tasks or chores are to be divided among agents. Although the definition of group-fairness and group envy-freeness can be extended seamlessly when there are either only chores or goods and chores, their relaxations do not.

In this paper, we take inspiration from the GF and GEF concepts and define several variants and relaxations of them that are well-defined for the more general setting of goods and chores. Our approach is similar in spirit to the work of Aziz, Caragiannis, Igarashi, and Walsh (2019b) who presented general definitions for fairness concepts that apply as well to the case of goods and for chores.

Contributions Our first conceptual contribution is to formalize relaxations of GEF for the case of goods and chores. We give general definitions which apply seamlessly to non-additive preference or even ordinal preferences. The main relaxation is group envy-freeness up to one item (GEF1), which we show to be incomparable to the relaxations of group-fairness (GF). We also introduce a stronger counterpart of GEF1, called s-GEF1, when groups of different size are allowed. We clarify the logical relations between these concepts through a clear taxonomy depicted in Figure 1.

We present two key existence and algorithmic results. First, we design a polynomial-time algorithm that always computes a GEF1 allocation for the case of identical preferences. The proof relies on interesting connections with two-sided matching and it invokes Hall's marriage theorem. We then focus on a natural class of mixed utilities called ternary symmetric utilities and design an algorithm that returns an allocation that satisfies GEF1. This algorithm involves network flows and makes use of several transformations of the utility functions. The results also provide additional insights on the connection between Nash social welfare and leximin welfare.

We then show that GEF1 allocations do not always exist for monotone utilities even with only goods. We also show that allocations satisfying the stronger concepts and group proportionality are not guaranteed to exist.

Finally, we prove that checking whether a given allocation satisfies GEF1 is coNP-complete when there are either only goods, only chores or both goods and chores.

2 Related Work

Fair division is a dynamic field in both economics and computer science (Young, 1995, Brams and Taylor, 1996, Moulin, 2004, Bouveret, Chevaleyre, and Maudet, 2016, Lang and Rothe, 2016, Moulin, 2018). It dates back to Steinhaus (1948) who introduced "the problem of fair division". This seminal work gave rise to an extensive literature focusing on how to allocate fairly and efficiently a set of item to a set of agents while taking into account the preferences of the agents. One of the most studied fairness concept is envy-freeness (Foley, 1967) which states that no agent would prefer having the share another agent received instead of his own.

Envy-freeness has been introduced in settings where items are divisible where it can always be satisfied. This no longer hold when considering indivisible items. However by relaxing envy-freeness so that no agent should envy another one if one good is removed from her share, called envy-freeness up to one good (EF1), one can guarantee existence of such criteria. This notion was implicit in Lipton et al. (2004)'s work and was formally introduced by Budish (2011). Similar relaxation but considering removing of any item called envy-freeness up to any good (EFX), has been introduced by Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang (2016). They moreover showed the compatibility between EF1 and Pareto-optimality: there always exists an allocation satisfying both.

Envy-freeness and its different relaxations are defined over pairs of agents —there should not be any envy between any two agents—, these concepts can be extended over groups of agents. Berliant et al. (1992) introduced the notion of *group envy-freeness* when items are infinitely divisible, extending the idea of *coalition fairness* (Vind, 1971, Schmeidler and Vind, 1972, Svensson, 1983) and *strict envy-freeness* (Zhou, 1992). According to Berliant et al. (1992), an allocation is group envy-free if for all pairs of groups of the same size S and T, there does not exist a reallocation of the items owned by agents in T to the agents in S that Pareto-dominates the current allocation. They proved existence of GEF allocations under some preference monotonicity assumptions and showed an equivalence between EF and GEF under specific assumptions. Their work has been extended later by Husseinov (2011) to *weak group envy-freeness*. Other works (Manurangsi and Suksompong, 2017, Halevi and Nitzan, 2016) considered allocating divisible items to pre-existent groups consisting of heterogeneous agents.

Similar extensions of envy-freeness over groups of agents have been proposed with indivisible items. Othman and Sandholm proposed a definition of EF for groups when monetary transfers between the agents are allowed. Later, Todo, Li, Hu, Mouri, Iwasaki, and Yokoo (2011) introduced *envy-freeness of a group toward a group* in the same setting. Aleksandrov and Walsh (2018) presented another definition of *group envy-freeness* between groups of potentially different sizes. Their definition, however, relies on interpersonal comparison which has received criticism in the social choice literature. They indeed use the arithmetical mean as a way to the aggregate utility for a group of agents. Conitzer et al. (2019) defined *group-fairness*, a definition similar to that of Berliant et al. (1992) but considering indivisible items and every possible pair of groups. They also introduced two different "up to one" relaxations of group-fairness for which they proved existence by using some variant of the Nash social welfare. Another line of work in the same spirit is to consider pre-existing groups of agents, taken as inputs of the procedures. Segal-Halevi and Suksompong (2018) introduced *democratic-fairness* and Kyropoulou, Suksompong, and Voudouris (2019) focused on envy-freeness and its relaxations in such settings. Similarly, Benabbou, Chakraborty, Elkind, and Zick (2019) investigated the problem of allocating indivisible goods to agents partitioned into types.

The chore division problem has been introduced by Gardner (1978) as a generalization of the fair division problem to divisible item negatively valued. Brams and Taylor (1996) and subsequent works (Su, 1999, Peterson and Su, 2002, Dehghani, Farhadi, HajiAghayi, and Yami, 2018) studied the *cake-cutting problem* when agents dislike the cake. Existence of envy-free allocations have been proved in this context. Segal-Halevi (2018) considered the mixed setting where the cake is composed of both desirable and undesirable parts. Extending these works, Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2016, 2017) studied the competitive division when there are multiple divisible goods and chores. They provided a generalization of the Gale-Eisenberg theorem (Gale, 1960, Eisenberg, 1961) for the case of goods and chores. Their work relies again on the Nash social welfare.

The fair division of chores has been recently studied in the setting of indivisible items. Caragiannis et al. (2012) analysed the impact of fairness on efficiency through the *price of fairness*. They showed several interesting differences between goods and chores settings for bounds on the price of fairness. Aziz, Rauchecker, Schryen, and Walsh (2017) and Barman and Krishnamurthy (2017) considered Max-min share with chores, showing non-existence and providing approximate algorithms for it. Aziz et al. (2019b) proposed a general framework for settings with both goods and chores. In particular, they provided a general definition for EF1 and presented an algorithm that computes an EF1 allocation even for arbitrary ordinal preferences over bundles of items.

3 Preliminaries

Let $\mathcal{N}$ be a set of n agents and $\mathcal{O}$ a set of m items. Agent $i \in \mathcal{N}$ has preferences over sets of items, called bundle, represented by a utility function $u_i : 2^{\mathcal{O}} \to \mathbb{R}$. We emphasize that agents can evaluate a bundle positively or negatively. Preferences are said to be *additive* if for every subset of items $O \subseteq \mathcal{O}$, we have $u_i(O) = \sum_{o \in O} u_i(o)$. We assume additive preferences throughout the paper except explicitly stated otherwise. Our definitions can be applied to non-additive preferences. An item $o \in \mathcal{O}$ is a good for i if $u_i(o) \geq 0$ and chore for i if $u_i(o) \leq 0$.

Let $O \subseteq \mathcal{O}$ be a subset of items and $N \subseteq \mathcal{N}$ a subset of agents. An a*llocation* $\pi = \left\langle \pi_1, \ldots, \pi_{|N|} \right\rangle$ over O and N is a vector of bundles $\pi_i \subseteq O$ for $i \in N$. It is such that:

1. an item can be allocated only to one person: $\forall i, j \in N$ s.t. $i \neq j, \pi_i \cap \pi_j = \emptyset$,
2. and all items are allocated: $\bigcup_{i \in N} \pi_i = O$.

For a subset of agents $N \subseteq \mathcal{N}$, we denote by $\pi_N = \bigcup_{i \in N} \pi_i$ the set of items held by agents in N. We write respectively π_i^+ and π_i^- the sets of goods and chores in π_i.

For a subset of items $O \subseteq \mathcal{O}$ and a subset of agents $N \subseteq \mathcal{N}$, we denote by $\Pi(O, N)$ the set of all the allocations over O and N. If $O \neq \mathcal{O}$, an allocation $\pi \in \Pi(O, \mathcal{N})$ is called *partial*. A triplet $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle$ is an instance. $\mathcal{I}$ is the set of all the instances, $\mathcal{I}^+$ the set of instances with only goods and $\mathcal{I}^-$ the set of instances with only chores.

Next, we give the definition if the efficiency concepts we will discuss in the paper.

Definition 1 (Pareto-optimality). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. Let $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ be an allocation, we say that an allocation π' Pareto-dominates π if all agents are better off in π' and at least one agent is strictly better off: $\forall i \in \mathcal{N}, u_i(\pi'_i) \geq u_i(\pi_i)$ and $\exists i \in \mathcal{N}, u_i(\pi'_i) > u_i(\pi_i)$.*

An allocation π is said to be Pareto-optimal (PO) if no other allocation Pareto-dominates it.

Definition 2 (Nash-optimality). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is said to be Nash-optimal if it maximizes the Nash social welfare: $\prod_{i \in \mathcal{N}} |u_i(\pi_i)|$.*

In the following, we introduce envy-freeness and its relaxations when dealing with goods and chores as presented by Aziz et al. (2019b).

Definition 3 (Envy-freeness). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is envy-free (EF) if and only if: $\forall i, j \in \mathcal{N}, u_i(\pi_i) \geq u_i(\pi_j)$.*

It is well known that for some instances, envy-free allocations does not exist, take for instance two agents and one item. To get around this impossibility, one can relax the definition of envy-freeness. Two different relaxations are usually considered, one existential (envy-freeness up to one item) and the other universal (envy-freeness up to any item).

Definition 4 (Envy-freeness up to one item). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is envy-free up to one item (EF1) if and only if:*

$$\forall i, j \in \mathcal{N}, \exists O \subseteq \pi_i \cup \pi_j, |O| \leq 1, \text{ s.t. } u_i(\pi_i \backslash O) \geq u_i(\pi_j \backslash O).$$

Definition 5 (Envy-freeness up to any item). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ satisfies envy-freeness up to any item (EFX) if and only if:*

$$\forall i, j \in \mathcal{N}, \begin{cases} \forall o \in \pi_i \text{ s.t. } u_i(\pi_i) - u_i(\pi_i \backslash \{o\}) < 0, & u_i(\pi_i \backslash \{o\}) \geq u_i(\pi_j) \\ \forall o \in \pi_j \text{ s.t. } u_i(\pi_j) - u_i(\pi_j \backslash \{o\}) > 0, & u_i(\pi_i) \geq u_i(\pi_j \backslash \{o\}). \end{cases}$$

Finally, we introduce the definition of proportionality, another very well studied fairness criteria.

Definition 6 (Proportionality (PROP)). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is proportional if and only if:*

$$\forall i \in \mathcal{N}, u_i(\pi_i) \geq \frac{u_i(\mathcal{O})}{n}.$$

Let us illustrate through an example the concepts we have introduced.

Example 1. *Consider the following instance with six items, $\mathcal{O} = \{o_1, \ldots, o_6\}$, and three agents, a_1, a_2 and a_3 whose preferences are additive. The utilities for the singletons are as follows.*

	o_1	o_2	o_3	o_4	o_5	o_6
a_1	$\boxed{2}$	1	−2	−3	$\boxed{-3}$	−4
a_2	−1	$\boxed{1}$	−1	$\boxed{0}$	2	−1
a_3	3	−6	$\boxed{-1}$	1	−4	$\boxed{-2}$

We call π the allocation represented by the squared items, that is $\pi_{a_1} = \{o_1, o_5\}$, $\pi_{a_2} = \{o_2, o_4\}$ and $\pi_{a_3} = \{o_3, o_6\}$. We have then the following utilities:

$$\begin{array}{lll} u_{a_1}(\pi_{a_1}) = -1 & u_{a_1}(\pi_{a_2}) = -2 & u_{a_1}(\pi_{a_3}) = -6 \\ u_{a_2}(\pi_{a_1}) = 1 & u_{a_2}(\pi_{a_2}) = 1 & u_{a_2}(\pi_{a_3}) = -2 \\ u_{a_3}(\pi_{a_1}) = -1 & u_{a_3}(\pi_{a_2}) = -5 & u_{a_3}(\pi_{a_3}) = -3 \end{array}$$

It appears then that agents a_1 and a_2 are not envious in π. However, agent a_3 is and would prefer having π_1 instead of π_3. The allocation π is therefore not envy-free. Nevertheless, by removing either o_3 from π_{a_3} or o_1 from π_{a_1}, agent a_3 would no longer be envious. Allocation π thus satisfies envy-free up to one good.

Observe moreover that the proportional shares, that is $u_i(\mathcal{O})/n$, for agents a_1, a_2 and a_3 are respectively -3, 0 and -3. Since every agent weakly exceed this threshold, allocation π is proportional.

4 Fairness criteria among groups of agents with goods and chores

In this section, we present our first contributions: a general definition for group envy-freeness and its relaxations in the presence of goods and chores. We then present a clear taxonomy of the different fairness criteria we discussed.

4.1 Group envy-freeness with goods and chores

We first introduce group envy-freeness, which is defined similarly when there are chores as when there are only goods.

Definition 7 (Group envy-freeness). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is GEF if for every $S, T \subseteq \mathcal{N}$ such that $|S| = |T| \neq 0$, there is no $\pi' \in \Pi(\pi_T, S)$, such that:*

$$\forall i \in S, \frac{|S|}{|T|} u_i(\pi'_i) \geq u_i(\pi_i),$$

with one inequality being strict.

We call the concept s-GEF if we do not impose the condition $|S| = |T|$.

In words, GEF states that there is no reallocation of π_T to the agents in S that would Pareto-dominates the current allocation for agents in S.

Note that s-GEF is equivalent to group-fairness (Conitzer et al., 2019). The name *group envy-freeness* is taken from Berliant et al. (1992) who introduced it for divisible items.

In the same spirit of EF1 and EFX, we introduce "up to one" and "up to any" relaxations for group envy-freeness.

Definition 8 (Group envy-freeness up to one item, GEF1). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is GEF1 if for every $S, T \subseteq \mathcal{N}$ where $|S| = |T| \neq 0$, for every $\pi' \in \Pi(\pi_T, S)$, and for every $i \in S$, there exists $O_i \subseteq \pi_i^- \cup \pi_i'^+$, $|O_i| \leq 1$, such that $\left\langle \frac{|S|}{|T|} u_i(\pi_i' \backslash O_i) \right\rangle_{i\in S}$ does not Pareto-dominate $\langle u_i(\pi_i \backslash O_i) \rangle_{i\in S}$.*

We talk about s-GEF1 if we do not impose the condition $|S| = |T|$.

Definition 9 (Group envy-freeness up to any item, GEFX). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is GEFX if for every $S, T \subseteq \mathcal{N}$ where $|S| = |T| \neq 0$, for every $\pi' \in \Pi(\pi_T, S)$, for every $i \in S$, for every $o_i \subseteq \pi_i^- \cup \pi_i'^+$, $\left\langle \frac{|S|}{|T|} u_i(\pi_i' \backslash \{o_i\}) \right\rangle_{i\in S}$ does not Pareto-dominate $\langle u_i(\pi_i \backslash \{o_i\}) \rangle_{i\in S}$.*

We call the concept s-GEFX if we do not impose the condition $|S| = |T|$.

Let us illustrate GEF1 on the instance presented in Example 1.

Example 2. *Coming back from the instance presented in Example 1, we claim that allocation π does not satisfy GEF1. Indeed, we can exhibit two groups of agents, S and T, and a reallocation $\pi' \in \Pi(\pi_T, S)$ such that every agent in S is better off in π' even after removing one item. Let us consider groups $S = \{a_2, a_3\}$ and $T = \{a_1, a_2\}$ and the reallocation π' such that $\pi'_{a_2} = \{o_2, o_4, o_5\}$ and $\pi'_{a_3} = \{o_1\}$. We have then:*

$$u_{a_2}(\pi'_{a_2} \backslash \{o_5\}) = 1 \geq u_{a_2}(\pi_{a_2}) = 1,$$
$$u_{a_3}(\pi'_{a_3} \backslash \{o_1\}) = 0 > u_{a_3}(\pi_{a_3}) = -3.$$

Reallocation π' is thus a Pareto-improvement for agents in S and S, T and π' are witnesses of a violation of GEF1 for π.

Before defining group proportionality, we clarify the relationships between GEF1 and the relaxations of group fairness. First, it should be noted that GEF1 and these relaxations are incomparable concepts: there exist allocations satisfying one concept but not the others.

Observe that with additive utility functions and for instances in $\mathcal{I}^+$, GEF1 is equivalent to *group fairness up to one good after* (GF1A) as defined by Conitzer et al. (2019). They also proposed *group fairness up to one good before* (GF1B) which is no longer relevant when there are chores since removing items cannot be done "before". GEF1 can be seen as an argument in favour of GF1A.

Nevertheless, s-GEF1 is not equivalent to GF1A even when considering only goods because of the way allocations are compared. Formally, $\langle u_i(\pi_i \cup \{o\}) \rangle_{i\in S}$ is compared to $\left\langle \frac{|S|}{|T|} u_i(\pi_i') \right\rangle_{i\in S}$ in GF1A while s-GEF1 compares $\langle u_i(\pi_i) \rangle_{i\in S}$ and $\left\langle \frac{|S|}{|T|} u_i(\pi_i' \backslash \{o\}) \right\rangle_{i\in S}$. The factor $\frac{|S|}{|T|}$ is then applied differently. GF1A seems to be specific to additive preferences while our intent is to define concepts that can conveniently be used for both additive and non-additive preferences.

Following this aim for a general definition that is suitable for general preference domains, we only consider groups of the same size to obtain ordinal properties. This is in the same spirit of envy-freeness and allows for more generality. It can also be argued that comparisons between same-sized groups implicitly captures comparisons between different sized groups: for a given k, one can compare the best subgroup in S of size k with the worst subgroup of T of size k.

4.2 Extending proportionality to groups of agents

When preferences are additive with either goods or chores, it is well known (see Aziz et al. (2019b) for example) that proportionality is a relaxation of envy-freeness. In a similar spirit, one can define *group proportionality*, a relaxation of GEF that extends proportionality to groups. It corresponds to GEF when T is fixed and set to $\mathcal{N}$. Note that it corresponds to the *core* as defined by Fain, Munagala, and Shah (2018) in the context of public good allocation.

Definition 10 (Group Proportionality)**.** *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}}\rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is group proportional (GP) if and only if for every $S \subseteq \mathcal{N}$, there is no $\pi' \in \Pi(\mathcal{O}, S)$, such that $\forall i \in S$, $\frac{|S|}{n} u_i\left(\pi'_i\right) \geq u_i(\pi_i)$, with at least one strict inequality.*

We can then define the usual relaxations, namely group proportionality up to one item (GP1) and up to any item (GPX).

Definition 11 (Group proportionality up to one item)**.** *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}}\rangle \in \mathcal{I}$ be an instance with both goods and chores. An allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ satisfies group proportionality up to one item (GP1) if and only if for every $S \subseteq \mathcal{N}$, for every allocation $\pi' \in \Pi(\mathcal{O}, S)$, for every $i \in S$, there exists $O_i \subseteq \pi_i^- \cup \pi_i'^+$, $|O_i| \leq 1$, such that $\left\langle \frac{|S|}{n} u_i(\pi'_i \backslash O_i)\right\rangle_{i\in S}$ does not Pareto-dominate $\langle u_i(\pi_i \backslash O_i)\rangle_{i\in S}$.*

Group proportionality up to any item can then be defined naturally using a universal quantifier instead of the existential one in the definition of GP1.

At this point the reader can be surprised that we did not introduce any relaxation of proportionality for only pairs of agents, PROP1 and PROPX. We did so because the definition given for setting with both goods and chores by Aziz et al. (2019b) does not correspond to the relaxation induced by GEF. For Aziz et al. (2019b), an allocation π satisfies PROP1 if for every agent $i \in \mathcal{N}$, we have: one of the following holds:

$$\exists O_i \subseteq \mathcal{O}, |O_i| \leq 1, \text{ s.t. } \begin{cases} u_i(\pi_i \cup O_i) \geq u_i(\mathcal{O})/n, or, \\ u_i(\pi_i \backslash O_i) \geq u_i(\mathcal{O})/n. \end{cases}$$

A definition closer to GP1, called PROP1′ in the following, would be to have for every agent i:

$$\exists O_i \subseteq \mathcal{O}, |O_i| \leq 1, \text{ s.t. } \begin{cases} u_i(\pi_i) \geq u_i(\mathcal{O} \backslash O_i)/n, or, \\ u_i(\pi_i \backslash O_i) \geq u_i(\mathcal{O})/n. \end{cases}$$

Both definitions have pros and cons. PROP1 is implied by EF1 which transmit the link between PROP and EF to their relaxations while PROP1′ and EF1 are incomparable. However, PROP1 is not implies by GP1 while PROP1′ is. Interestingly, one can observe that PROP1 is similar to GF1A in the way the group factor $|S|/|T|$ is applied (see the discussion comparing GEF1 and GF1A above).

4.3 Taxonomy of some fairness criteria

We present in Figure 1 a taxonomy of the different criteria discussed before. The links between s-GEF, GEF, GP and their relaxations are immediately derived from the definitions. Envy-freeness concepts are implied by GEF and s-GEF when S and T are singletons. GP implies PROP when S is a singleton. PO is implied by s-GEF, GEF and GP for $S = T = \mathcal{N}$.

For the existence results, allocations satisfying either PO or EF1 are guaranteed to exist (Aziz et al., 2019b). The absence of guarantee to find allocations satisfying EF, PROP, GEF, GP and s-GEF comes from the painting example discussed in the introduction. Negative results for the existence of allocations satisfying GP1, GPX, s-GEF1 and s-GEFX are presented in Section 7. No result is known about the existence of EFX allocations, even for the case of only good (see Plaut and Roughgarden (2018) for recent results). Finally there is currently no result of existence of GEF1 or GEFX allocations when preferences are additive and with goods and chores.

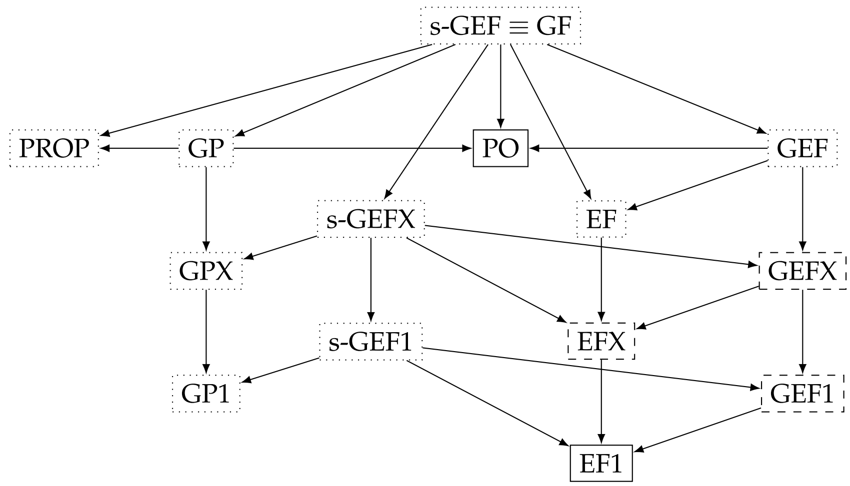

Figure 1: Logical relationship between fairness and efficiency criteria. Allocations satisfying concepts in dotted (resp. plain) are not (resp. are) guaranteed to exist. No existence result is known for concepts in dashed.

To conclude this section, we show through a simple example that although GEF implies Pareto-optimality and envy-freeness, GEF1 is more stringent than the combination of these two criteria. The following example illustrates that even on very restricted preference domain, PO and EF do not imply GEF1.

Example 3. *Consider the following instance with eight items, from o_1 to o_8, and four agents, a_1, a_2, a_3, a_4 whose preferences are additive and single-peaked with respect to the axis $\langle o_1, \ldots, o_8 \rangle$. The utilities for the singletons are as follows.*

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
a_1	-1	-1	1	1	0	0	0	0
a_2	0	0	0	0	1	1	-1	-1
a_3	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	0	0	0	0
a_4	0	0	0	0	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$

We call π the allocation represented by the squared items. π is clearly envy-free: agents a_3 and a_4 have their maximal utility and their bundles give 0 utility to agents a_1 and a_2. The allocation is moreover Pareto-optimal. However, $S = \langle a_1, a_2 \rangle$, $T = \langle a_3, a_4 \rangle$, $\pi'_{a_1} = \{o_3, o_4, o_7, o_8\}$ and $\pi'_{a_2} = \{o_1, o_2, o_5, o_6\}$, are witnesses of a violation of GEF1. We have:

$$u_{a_1}(\{o_3, o_4, o_7, o_8\} \setminus \{o_4\}) = 1 > u_{a_1}(\pi_{a_1}) = 0,$$
$$u_{a_2}(\{o_1, o_2, o_5, o_6\} \setminus \{o_4\}) = 1 > u_{a_2}(\pi_{a_2}) = 0.$$

Agents in S are then better off in π' even after removing one item.

5 The Egal-Sequential Algorithm for Identical Utilities

In this section, we present the Egal-Sequential Algorithm that returns a GEF1 allocation when preferences are identical. Identical preferences constitute an important and natural class of preferences especially if the item's values are objective or publicly known. The algorithm allocates sequentially the items in decreasing order of absolute utility. The item to be allocated is given to the worse off agent

Algorithm 1: The Egal-Sequential Algorithm

Input: An instance $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}}) \rangle$ with identical utility function u
Output: $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ an allocation satisfying EFX and GEF1

Set π to the empty allocation
Order items $o_1, \ldots, o_m$ in $\mathcal{O}$ in decreasing order of $|u(o)|$
for $j = 1$ *to* m **do**
  if $u(o_j) \geq 0$ **then**
    Choose $i^* \in \arg\min_{i \in \mathcal{N}} u(\pi_i)$
  else
    Choose $i^* \in \arg\max_{i \in \mathcal{N}} u(\pi_i)$
  Give o_j to i^*: $\pi_{i^*} \leftarrow \pi_{i^*} \cup \{o_j\}$
return π

if it is a good and to the better off agent otherwise. We prove that this ensures GEF1 in two steps. We first show that the Egal-Sequential Algorithm returns an EFX allocation and then that every EFX allocation is also GEF1 when preferences are identical.

Before moving to the proof, we define formally what we mean by identical preferences.

Definition 12 (Identical Preferences)**.** *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. Preferences $(u_i)_{i \in \mathcal{N}}$ are said to be identical if they are additive and there exists a utility function u such that $\forall i \in \mathcal{N}, \forall o \in \mathcal{O}, u_i(o) = u(o)$.*

We refer to u as the common utility function.

Our first lemma states that for identical preferences, the Egal-Sequential Algorithm present as Algorithm 1 returns allocations satisfying EFX.

Lemma 1. *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}}) \rangle \in \mathcal{I}$ be an instance, if $(u_i)_{i \in \mathcal{N}}$ is a profile of identical utility functions then the Egal-Sequential Algorithm returns an allocation π that satisfies EFX in time in $\mathcal{O}(\max\{m \log m, mn\})$.*

Proof. We call u the common utility function. We show by induction on the number of items that have already been allocated that the Egal-Sequential Algorithm maintains a partial allocation that is EFX. Let us denote by π^k the partial allocation constructed after allocating the k-th item.

The base case for $k = 1$ is straightforward. EFX is trivially satisfied by π^1 as only one item has been allocated.

Let us now suppose that for a given $k < m$ the partial allocation π^k is EFX. We show that π^{k+1} also satisfies EFX. Let o be the item that is allocated to agent i^* at the $k+1$-th step of the algorithm. Let us distinguish two cases depending on whether o is a good or a chore.

- If $u(o) \geq 0$, the only change in the utilities of the agents is an increase of i^*'s utility. The only possibility to violate EFX would therefore be for agents to become envious up to any item of i^*. As $i^* \in \arg\min_{i \in \mathcal{N}} u(\pi_i^k)$, no agent envies i^* in π^k: $\forall j \in \mathcal{N}, u(\pi_{i^*}^k) \leq u(\pi_j^k)$. This implies that $\forall j \in \mathcal{N}, u(\pi_{i^*}^{k+1} \backslash \{o\}) \leq u(\pi_j^{k+1})$. Since o is the smallest item allocated at step k, no agent envies i^* up to any item in π^{k+1}. Hence, π^{k+1} is EFX.
- If $u(o) < 0$, the only change in the utilities is a decrease of i^*'s utility. Therefore, the only possibility for EFX to be violated would be if i^* becomes envious up to any item of another agent. Since $i^* \in \arg\max_{i \in \mathcal{N}} u(\pi_i^k)$, i^* does not envy anyone in π^k, that is, $\forall j \in \mathcal{N}, u(\pi_{i^*}^k) \geq u(\pi_j^k)$. This implies that $\forall j \in \mathcal{N}, u(\pi_{i^*}^{k+1} \backslash \{o\}) \geq u(\pi_j^{k+1})$. Thus, i^* is still not envious since o is the smallest item allocated. Hence, π^{k+1} is still EFX.

We have then proved that the Egal-Sequential Algorithm returns EFX allocations when preferences are identical.

Sorting items can be done in time in $\mathcal{O}(m \log m)$. The for loop of the algorithm uses m steps during which finding an agent with maximum or minimum utility can be done in time in $\mathcal{O}(n)$ hence the overall complexity is $\mathcal{O}(\max\{m \log m, mn\})$. $\square$

To achieve GEF1, we show in the following that with identical preferences, any allocation satisfying EFX also satisfies GEF1.

Lemma 2. *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance. If $(u_i)_{i\in\mathcal{N}}$ is a profile of identical utility functions, then any allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ which is EFX is also GEF1.*

Proof. Let us consider an allocation π that satisfies EFX but not GEF1. Since preferences are identical we can assume, without loss of generality, that no item gives zero utility. If such items exist, they can be allocated to any agent without changing anything. As π is not GEF1, there exist two groups $S \subseteq \mathcal{N}$ and $T \subseteq \mathcal{N}$ and a reallocation $\pi' \in \Pi(\pi_T, S)$ such that $\forall i \in S, \forall o \in \pi_i^- \cup \pi_i^+, u(\pi'_i \backslash \{o\}) \geq u(\pi_i \backslash \{o\})$ with one inequality being strict. For $i \in S$, we introduce s_i defined as:

$$s_i = \max\{\max_{o\in\pi_i'^+} u(o), \max_{o\in\pi_i^-} -u(o), 0\}.$$

Violating GEF1 then implies: $\forall i \in S, u(\pi'_i) - u(\pi_i) \geq s_i$ with one inequality being strict. The 0 component in the definition of s_i is meant to tackle the case when an agent i receives an empty allocation in π and no goods in π'. If $\pi_i = \pi_i'^+ = \emptyset$ we should have $\pi_i^- = \emptyset$ to get a GEF1 violation, hence we have $u(\pi_i) = u(\pi'_i)$ and $s_i = 0$ is a suitable bound.

By summing the inequalities coming from the violation of GEF1 over $i \in S$, we obtain $\sum_{i\in S} u(\pi'_i) - \sum_{i\in S} u(\pi_i) > \sum_{i\in S} s_i$. Since preferences are identical and additive this implies:

$$u(\pi_T) - u(\pi_S) > \sum_{i\in S} s_i. \tag{1}$$

Moreover, as π is EFX, we have $\forall i, j \in \mathcal{N}, \forall o \in \pi_i^- \cup \pi_j^+, u(\pi_i \backslash \{o\}) \geq u(\pi_j \backslash \{o\})$. For $i, j \in \mathcal{N}$, we introduce $s_{i,j}$ defined by:

$$s_{i,j} = -\min\{\min_{o\in\pi_j^+} -u(o), \min_{o\in\pi_i^-} u(o), 0\}.$$

The 0 component is once again here to take care of the case when $\pi_i^- \cup \pi_j^+ = \emptyset$. We have thus:

$$\forall i, j \in \mathcal{N}, s_{i,j} \geq u(\pi_j) - u(\pi_i). \tag{2}$$

Our goal is now to sum up inequalities describe in (2) to obtain a contradiction with (1). We are looking for a set of pairs (i, j) such that each $i \in S$ and each $j \in T$ appear once and only once, and such that the sum of $s_{i,j}$ over these pairs is smaller than $\sum_{i\in S} s_i$. To do so, we find a suitable matching in a bipartite graph (see Roth and Sotomayor (1992) for general introduction on the topic).

Let us consider the bipartite graph $G = \langle S \cup T, E \rangle$ where nodes represent agents in S and in T. There is an edge $(i, j) \in E$ between agents $i \in S$ and $j \in T$ if and only if $\pi_i'^+ \cap \pi_j \neq \emptyset$, that is, i receives some of j's goods in π'. We consider a partition $S^+ \cup S^-$ of S where:

$$S^+ = \{i \in S \mid u(\pi_i) \geq 0\}$$
$$S^- = \{i \in S \mid u(\pi_i) < 0\}.$$

For $X \subseteq S$, we write $N(X)$ its neighbourhood in the graph G: $N(X) = \{j \in T \mid \exists i \in X, (i, j) \in E\}$. A symmetric definition holds for $N(Y)$ where $Y \subseteq T$.

We claim that there always exists a matching $M \subseteq S \times T$ in G that matches all the agents in S^+. Suppose for the sake of contradiction that such M does not exist. From Hall's theorem (Hall, 1935),

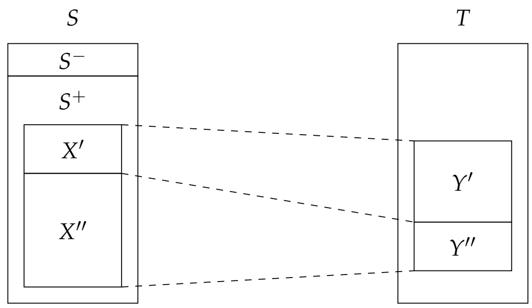

Figure 2: Sets of agents considered in the proof of Lemma 2.

there must exist $X \subseteq S^+$ and $Y \subseteq T$ such that $Y = N(X)$ and $|Y| < |X|$. Let us assume that X is a smallest Hall's violation in G, and consider $M' \subseteq X \times Y$ a maximum matching between agents in X and in Y.

We show in the following that M' always match all the agents in Y. To get a better understanding, a diagram illustrating the different set of agents considered is presented in Figure 2. Assume that M' does no match all the agents in Y, Hall's theorem implies that there exist $X' \subseteq X$ and $Y' \subseteq Y$ such that $X' = N(Y')$ and $|X'| < |Y'|$. Then for $X'' = X \setminus X'$ and $Y'' = Y \setminus Y'$, we have $|X''| > |Y''|$ and $N(X'') = Y''$. Indeed, as $|X| > |Y|$ we have $|X| - |X'| > |Y| - |X'|$ and thus $|X''| > |Y''|$ since $|X'| < |Y'|$. Moreover it is clear that $N(X'') = Y''$ as otherwise any agent in X'' linked to another agent in Y' is in $N(Y') = X'$ and can not be in X''. Overall, X'' constitutes a Hall's violation for the existence of M which is smaller than X. This contradicts the fact that X is the smallest such Hall's violation. Sets X' and Y' does not exist and M' matches all the agents in Y.

We now turn back to the problem of showing that M matches all the agents in S^+. As M' matches all the agents in Y and u is additive, we have $\sum_{(i,j)\in M'} u(\pi_j) = u(\pi_Y)$. Hence, by summing inequalities (2) for $(i,j) \in M'$, we obtain:

$$u(\pi_Y) - \sum_{(i,j)\in M'} s_{i,j} \leq \sum_{(i,j)\in M'} u(\pi_i) \tag{3}$$

Moreover summing inequalities yielded by the violation of GEF1 over $i \in S$ in a similar manner as inequalities (1) brings $u(\pi_Y) - \sum_{i\in X} s_i \geq u(\pi_X)$. Together with (3), this leads to:

$$\sum_{(i,j)\in M'} u(\pi_i) + \sum_{(i,j)\in M'} s_{i,j} \geq u(\pi_Y) \geq u(\pi_X) + \sum_{i\in X} s_i. \tag{4}$$

Observe that for any pair of agents (i,j) such that there is an edge between i and j in G, we have:

$$s_i \geq \max_{o\in\pi_i'^+} u(o) \geq \max_{o\in\pi_i'^+\cap\pi_j} u(o) \geq \min_{o\in\pi_j^+} u(o) \geq s_{i,j}. \tag{5}$$

In addition, as $|M'| < |X|$, it is clear that $\sum_{(i,j)\in M'} u(\pi_i) < \sum_{i\in X} u(\pi_i)$ and thus $\sum_{(i,j)\in M'} u(\pi_i) < u(\pi_X)$. Overall, we have:

$$\sum_{(i,j)\in M'} u(\pi_i) + \sum_{(i,j)\in M'} s_{i,j} < u(\pi_X) + \sum_{i\in X} s_i,$$

which contradicts (4). Therefore, no Hall's violation can exist and we have proved that the matching M does match all the agents in S^+.

Next we show that the existence of this matching M leads to a contradiction on the fact that π is not GEF1. We extend the matching M to match all the agents in S by arbitrarily pairing each agent

$i \in S^-$ with an unmatched agent in T. Let the extended matching be called M^*. Observe that for any new pair of agents $(i,k) \in M^* \setminus M$, we have $i \in S^-$, that is $\pi_i^- \neq \emptyset$. Hence, for any agent $j \in T$, we have:

$$s_{i,j} \leq \min_{o \in \pi_i^-} u(o) \leq \max_{o \in \pi_i^-} u(o) \leq s_i. \tag{6}$$

By summing (2) over $(i,j) \in M^*$ we obtain $\sum_{(i,j)\in M^*} s_{i,j} \geq u(\pi_T) - u(\pi_S)$. From (5) and (6) we get that $\sum_{(i,j)\in M^*} s_{i,j} \leq \sum_{i \in S} s_i$, hence summing (2) over $(i,j) \in M^*$ leads to a contradiction with (1). We have thus proved that π satisfies both EFX and GEF1. □

A direct consequence of the two previous lemmas is that a GEF1 allocation can be computed by the Egal-Sequential Algorithm. The proof just amounts at linking Lemmas 1 and 2.

Theorem 1. *For identical preferences, an allocation satisfying GEF1 always exists and can be computed in linear time by the Egal-Sequential Algorithm.*

Conitzer et al. (2019) showed that when preferences are identical their relaxation of group-fairness is implied by EFX. We significantly extend their result in different ways. Firstly, our result applies in the case of mixed utilities where preferences can model both goods and chores. Secondly, we provide a linear time algorithm to compute GEF1 allocation with identical preferences. Finally, our proof does not involve Nash-optimality which is not a suitable solution concept with chores as we will show below.

6 The Ternary Flow Algorithm

In this section, we focus on another restriction of the preferences, namely *ternary symmetric preferences*. We provide an algorithm that computes GEF1 allocations for ternary symmetric preferences. We do so by proving that any leximin-optimal allocation is also GEF1 and by providing an algorithm returning a leximin-optimal allocation in polynomial time. Similar links between leximin-optimality and envy-freeness concepts have been observed by Plaut and Roughgarden (2018) for the case of goods.

Definition 13 (Ternary symmetric preferences). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with both goods and chores. Preferences $(u_i)_{i\in\mathcal{N}}$ are said to be ternary symmetric if they are additive and for every agent $i \in \mathcal{N}$, there exists $\alpha_i \in \mathbb{R}_{\geq 0}$ such that: $\forall o \in \mathcal{O}, u_i(o) \in \{-\alpha_i, 0, \alpha_i\}$.*

We first provide a characterization of Pareto-optimality for ternary symmetric utilities.

Lemma 3. *Let $\mathcal{I} = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle$ be an instance where $(u_i)_{i\in\mathcal{N}}$ describes a profile of ternary symmetric utility functions. An allocation $\pi \in \Pi(\mathcal{N}, \mathcal{O})$ is Pareto-optimal if and only if for every item $o \in \mathcal{O}$ we have:*

$$\begin{cases} o \in \pi_i \text{ with } u_i(o) > 0, \text{iff } \max_{j\in\mathcal{N}} u_j(o) > 0, \\ o \in \pi_i \text{ with } u_i(o) = 0, \text{iff } \max_{j\in\mathcal{N}} u_j(o) = 0, \\ o \in \pi_i \text{ with } u_i(o) < 0, \text{iff } \max_{j\in\mathcal{N}} u_j(o) < 0. \end{cases}$$

Proof. Since Pareto-optimality is invariant under rescaling of utilities, assume w.l.o.g. that $\alpha_i = 1$, $\forall i \in \mathcal{N}$. Then, if the three conditions hold, every item is allocated to an agent having maximal utility for it. This implies Pareto-optimality.

Next, assume that one item o is allocated to an agent i who do not have maximal utility for it: o is a chore (resp. neutral) for i and there is j who considers o as either neutral or a good (resp. a good). Then transferring o from i to j leads to a Pareto-improvement. □

For a profile of ternary symmetric preferences $(u_i)_{i\in\mathcal{N}}$, we introduce the normalized profile, written $(u_i^{\text{Norm}})_{i\in\mathcal{N}}$ and defined as:

$$\forall i \in \mathcal{N}, \forall o \in \mathcal{O}, u_i^{\text{Norm}}(o) = \begin{cases} 1 & \text{if } u_i(o) > 0 \\ 0 & \text{if } u_i(o) = 0 \\ -1 & \text{if } u_i(o) < 0 \end{cases}$$

Such preferences model statements such as "I like", "I am indifferent" and "I do not like". It is close to the idea of approval and disapproval voting (Brams and Fishburn, 1978, Felsenthal, 1989). In the fair division literature, it has also been referred to as dichotomous preferences when there are only goods (Bogomolnaia, Moulin, and Stong, 2005).

Next we introduce leximin optimality. For an allocation π we denote by $\vec{u}(\pi) \in \mathbb{R}^n$ the vector of the utilities in π sorted increasingly. For two vectors $\vec{u}, \vec{v} \in \mathbb{R}^k$, we say that $\vec{u}$ leximin-dominates $\vec{v}$, written $\vec{u} \succ_{lex} \vec{v}$, if there exits an index $i \leq k$ such that $\vec{u}_j = \vec{v}_j, \forall j < i$, and $\vec{u}_i > \vec{v}_i$. Finally, an allocation π is said to be leximin-optimal if there is no allocation π' such that $\vec{u}(\pi) \succ_{lex} \vec{u}(\pi')$.

The algorithm we use to compute leximin-optimal allocations uses cost flow network. For details on network flows, we refer the reader to Schrijver (2003, chapters 10 and 12). Let us introduce it briefly. A cost flow network is represented by a directed graph $G = \langle V, E\rangle$ whose nodes $v \in V$ are labelled with a demand $d(v) \in \mathbb{R}$ while edges $e \in E$ are labelled by a maximum capacity $\delta(e) \in \mathbb{R}_{>0}$ and a cost $c(e) \in \mathbb{R}_{\geq 0}$. A flow f is a mapping $f : E \to \mathbb{R}_{\geq 0}$ where $f(e)$ is the amount of flow passing through the edge e. By a slight abuse of notation, we use $f(v)$ for a node $v \in V$ to denote the difference between the outgoing flow and the incoming flow in v. A flow is realizable if for every edge $e \in E$ its capacity is not exceeded, $f(e) \leq \delta(e)$, and for every node $v \in V$ its demand is satiated, $f(v) = d(v)$. The cost of a realizable flow f is defined by $c(f) = \sum_{e\in E} f(e)c(e)$. A minimum cost flow is then a flow with minimal cost among the realizable flows. A flow is said to be integer if $\forall e \in E, f(e) \in \mathbb{N}$.

Next, we present the Nash Flow Algorithm proposed by Darmann and Schauer (2015). Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}}\rangle \in \mathcal{I}^+$ be an instance with only goods and where preferences are normalized ternary symmetric. Note that since there are only goods, the utilities for the singletons are in $\{0, 1\}$. The network $G = \langle V, E\rangle$ consists in the set of nodes $V = \{s, t\} \cup \mathcal{N} \cup \mathcal{O} \cup \{t_{i,j} \mid i \in \mathcal{N}, o_j \in \mathcal{O}\}$ where the demands are $d(s) = m$, $d(t) = -m$ and $d(v) = 0$ for every $v \in V\backslash\{s, t\}$. The edge set E is defined by:

- for every item $o \in \mathcal{O}$, the edge (s, o) is in E with capacity $\delta(s, o) = 1$ and cost $c(s, o) = 0$,
- for every agent $i \in \mathcal{N}$, for every item $o \in \mathcal{O}$ such that $u_i(o) = 1$, the edge (o, i) is in E with capacity $\delta(o, i) = 1$ and cost $c(o, i) = 0$,
- for every agent $i \in \mathcal{N}$ and every item $o_j \in \mathcal{O}$, two edges are in E: $(i, t_{i,j})$ with capacity $\delta(i, t_{i,j}) = 1$ and cost $c(i, t_{i,j}) = n^j$ and $(t_{i,j}, t)$ with capacity $\delta(t_{i,j}, t) = 1$ and cost $c(t_{i,j}, t) = 0$. In the following we refer to $(i, t_{i,j})$ edges as "t" edges.

Since the capacities are all integer, there always exists a minimum cost integer flow of G by the integrality property. We have moreover a one-to-one correspondence between integer flows in G and allocations in $\Pi(\mathcal{N}, \mathcal{O})$. From a flow f, an allocation π is defined by $o \in \pi_i$ if and only if $f(o, i) = 1$. From an allocation π the flow f is defined by $f(s, o) = 1$ for each $o \in \mathcal{O}$, $f(o, i) = 1$ if and only if $o \in \pi_i$ for every $o \in \mathcal{O}$ and $i \in \mathcal{N}$, and $f(i, t_{i,j}) = f(t_{i,j}, t) = 1$ for every $i \in \mathcal{N}$ and $1 \leq j \leq u_i(\pi_i)$. An allocation π is Nash-optimal if and only if it corresponds to a minimum cost integer flow of the network G.

We first extend Darmann and Schauer (2015)'s result by showing that the allocation returned by the Nash Flow Algorithm is also leximin-optimal.

Lemma 4. *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}}\rangle \in \mathcal{I}^+$ be an instance with only goods and where $(u_i)_{i\in\mathcal{N}}$ describes a profile of normalized ternary utility functions. An allocation π is leximin-optimal if and only if it correspond to a minimum cost integer flow in the network defined by the Nash Flow Algorithm.*

Algorithm 2: The Ternary Flow Algorithm

Input: An instance $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}}) \rangle$ such that $\forall i \in \mathcal{N}, u_i = u$, for a given utility function u

Output: $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ a GEF1 allocation

Set O^+ to $\{o \in \mathcal{O} : \max_{i \in \mathcal{N}} u_i(o) > 0\}$

Set O^0 to $\{o \in \mathcal{O} : \max_{i \in \mathcal{N}} u_i(o) = 0\}$

Set O^- to $\{o \in \mathcal{O} : \max_{i \in \mathcal{N}} u_i(o) < 0\}$

Consider new utility functions $(u'_i)_{i \in \mathcal{N}}$ such that

$$\forall i \in \mathcal{N}, \forall o \in O^+, u'_i(o) = \begin{cases} 1 & \text{if } u_i^{\text{Norm}}(o) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Run the Nash Flow Algorithm on $I' = \langle \mathcal{N}, O^+, (u'_i)_{i \in \mathcal{N}} \rangle$ to obtain the partial allocation π

for $o \in O^-$ **do**

  Allocate o to $i^* \in \arg\max_{i \in \mathcal{N}} u_i^{\text{Norm}}(\pi_i)$ and update π

for $o \in O^0$ **do**

  Allocate o to $i^* \in \arg\min_{i \in \mathcal{N}, u_i^{\text{Norm}}(o) = 0} u_i^{\text{Norm}}(\pi_i)$ and update π

return π

Proof. For a flow f, we denote by f_i, $i \in \mathcal{N}$, the amount of flow passing through agent node i, that is $f_i = \sum_{j \in \mathcal{O}} f(j, i)$. Note that f_i is then the number of good received by agent i in the allocation corresponding to f. Since utilities are normalized ternary, it is also i's utility, $u_i(\pi_i)$.

We first show that a leximin-optimal allocation π corresponds to a minimum cost flow f. Let us suppose toward a contradiction that there exists two agents i_1 and i_2 with $f_{i_1} - f_{i_2} \geq 2$ and a good o such that $(o, i_2) \in E$ and $f(o, i_2) = 1$. We have thus $u_{i_1}(o) = u_{i_2}(o) = 1$. Consider then the allocation π' such that $\pi'_k = \pi_k, \forall k \in \mathcal{N} \setminus \{i_1, i_2\}$, $\pi'_{i_1} = \pi_{i_1} \setminus \{o\}$ and $\pi'_{i_2} = \pi_{i_2} \cup \{o\}$ and denote by f' the corresponding flow. Since $f_{i_1} - f_{i_2} \geq 2$, agent i_2 comes before agent i_1 in the lexmin ordering of the utilities in π. It is also the case in π' since $f'_{i_1} - f'_{i_2} = f_{i_1} - f_{i_2} - 2$. Moreover we have $u_{i_2}(\pi'_{i_2}) > u_{i_2}(\pi_{i_2})$, hence $\vec{u}(\pi') \succ_{lex} \vec{u}(\pi)$ which is a contradiction. Agents i_1 and i_2 do not exist which is a sufficient condition[1] for f to be a minimum cost flow.

Next we show the other implication. For a flow f, we write $\vec{f}$ the vector of the f_i ordered from the lowest to the highest. Observe that $\vec{f} = \vec{u}(\pi)$ for π the allocation corresponding to f. Darmann and Schauer (2015, proof of Theorem 3.3, step 3) proved that all minimum cost integer flows have the same vector $\vec{f}$. We just proved that there exists a minimum cost integer flow f^* corresponding to a leximin-optimal allocation π^*, hence any minimum cost integer flow f is such that $\vec{f} = \vec{f^*}$, which proves that the allocation π corresponding to f is leximin-optimal. □

This equivalence implies that any leximin-optimal allocation also maximizes the Nash social welfare.

Corollary 1. *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}^+$ be an instance with only goods and where $(u_i)_{i \in \mathcal{N}}$ describes a profile of normalized ternary utility functions. Any leximin-optimal allocation π is Nash-optimal.*

Making use of the Nash Flow Algorithm, we propose the Ternary Flow Algorithm (Algorithm 2). It computes a leximin-optimal allocation on the normalized utilities which corresponds to a GEF1 allocation with respect to the original preferences.

[1] Darmann and Schauer (2015, proof of Theorem 3.3, step 1) showed that an integer flow f is a minimum cost flow if and only if $\forall i \in \mathcal{N}, \forall h$ s.t. $1 \leq h \leq f_i, f(i, t_{i,h}) = 1$ and there is no sequence $\langle i_1, o_1, i_2, o_2, \dots, o_{l-1}, i_l \rangle$ with $f_{i_1} - f_{i_l}$ such that for all $1 \leq h \leq l-1$, we have $(o_h, i_{h+1}) \in E$ and $f(o_h, i_h) = 1$.

Lemma 5. *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i\in\mathcal{N}} \rangle \in \mathcal{I}$ be an instance with goods and chores and where $(u_i)_{i\in\mathcal{N}}$ describes a profile of ternary symmetric preferences. The allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ returned by the Ternary Flow Algorithm is leximin-optimal for the normalized preferences $(u_i^{Norm})_{i\in\mathcal{N}}$.*

Proof. We show that along the execution of Algorithm 2, the partial allocation π is always leximin-optimal.

Based on Lemma 4, we know that after line 5 of Algorithm 2, π is leximin-optimal for the utility profile $(u'_i)_{i\in\mathcal{N}}$. Since agents are only allocated items they have utility 1 for, π is also leximin-optimal for $(u_i^{\text{Norm}})_{i\in\mathcal{N}}$.

Next, we show by induction that π is always leximin-optimal according to the normalized utilities throughout the first "for loop" (line 6) where items in O^- are allocated. In the following, leximin-optimality is according to the normalized utilities. Consider one step of the loop, we call $o \in O^-$ the chore that is to be allocated, μ the current allocation, that is leximin-optimal, and π the allocation where o has been allocated to agent $i^* \in \arg\max_{i\in\mathcal{N}} u_i^{\text{Norm}}(\pi_i)$. In the following, we use $\vec{u}$ to refer to the leximin ordering according to preferences $(u_i^{\text{Norm}})_{i\in\mathcal{N}}$. Suppose that π is not leximin-optimal, then there exist an allocation π' so that there exists an integer $k \in [\![1, n]\!]$ such that $\vec{u}_j(\pi') = \vec{u}_j(\pi), \forall j < k$, and $\vec{u}_k(\pi') > \vec{u}_k(\pi)$. Since $i^* \in \arg\max_{i\in\mathcal{N}} u_i^{\text{Norm}}(\pi_i)$ we can assume without loss of generality that i^*'s index in $\vec{u}(\pi)$ is n. From the induction hypothesis we know that μ is leximin-optimal, hence $n \leq k$ as otherwise the allocation obtained from π' by deleting chore o would leximin-dominates μ. We have thus $k = n$, that is $\sum_{i\in\mathcal{N}} \vec{u}_i(\pi') > \sum_{i\in\mathcal{N}} \vec{u}_i(\pi)$. However since o is a chore for every agent and μ is leximin-optimal, we know from Lemma 3 that π is Pareto-optimal. Moreover, $\pi'$is also Pareto-optimal, since it is leximin-optimal, Lemma 3 implies then $\sum_{i\in\mathcal{N}} \vec{u}_i(\pi) = \sum_{i\in\mathcal{N}} \vec{u}_i(\pi')$. This is a contraction. We have then proved that the allocation π is still leximin-optimal after allocating chore o. The induction is settled and after the first "for loop", the partial allocation π is leximin-optimal.

Finally, note that the items in O^0 are allocated to agents that have 0 utility for them. Hence, the allocation π is still leximin-optimal after the second "for loop". After this last loop, π is no longer a partial allocation, all the items have been allocated. Hence, Algorithm 2 returns π that is leximin-optimal according to the normalized utilities. $\square$

Lemma 6. *If the preferences are normalised ternary symmetric, any leximin-optimal allocation also satisfies GEF1.*

Proof. In the following we consider an allocation π, supposed to be leximin-optimal. We first show that the agents' utility can not be too different because of leximin-optimality.

Claim 1. *For every $i, j \in \mathcal{N}$, if $u_j(\pi_j) - u_i(\pi_i) \geq 2$ then we have:*

$$\{o \in \pi_j \mid u_i(o) = 1\} \cup \{o \in \pi_i \mid u_j(o) = -1\} = \emptyset.$$

Proof: Assume that there is an item $o \in \pi_j$ such that $u_i(o) = 1$. Consider the allocation π' such that $\forall k \in \mathcal{N} \backslash \{i, j\}, \pi'_k = \pi_k, \pi'_i = \pi_i \cup \{o\}, \pi'_j = \pi_j \backslash \{o\}$. Since $u_j(\pi_j) - u_i(\pi_i) \geq 2$, i appears before j in $\vec{u}(\pi)$. Moreover, as $u_j(\pi'_j) - u_i(\pi'_i) \geq 0$, i also appears before j in $\vec{u}(\pi')$. However, $u_i(\pi'_i) > u_i(\pi_i)$ implies that $\vec{u}(\pi') \succ_{lex} \vec{u}(\pi)$ which contradicts the fact that π is leximin-optimal. The case where there exists an item $o \in \pi_i$ such that $u_j(o) = -1$ is exactly symmetric. $\blacksquare$

Next, we show our second claim that if one agent has negative utility for her bundle, then all agent's utility is at most one more.

Claim 2. *If $min_{i\in\mathcal{N}} u_i(\pi_i) < 0$, then $\forall i \in \mathcal{N}, 0 \leq u_i(\pi_i) - min_{i\in\mathcal{N}} u_i(\pi_i) \leq 1$.*

Proof: Let $k \in \mathcal{N}$ be the agent with minimum utility in π. Assume that $u_k(\pi_k) < 0$, it means that k owns at least one chore o in π. Since π is Pareto-efficient, from Lemma 3 we know that o is a chore for every agent in $\mathcal{N}$. Let us assume that there exists an agent $j \in \mathcal{N}$ such that $u_j(\pi_j) > u_k(\pi_k) + 1$, then since o is also a chore for j, the allocation obtained by transferring o from k to j would leximin-dominates π. This would contradict the leximin-optimality of π. $\blacksquare$

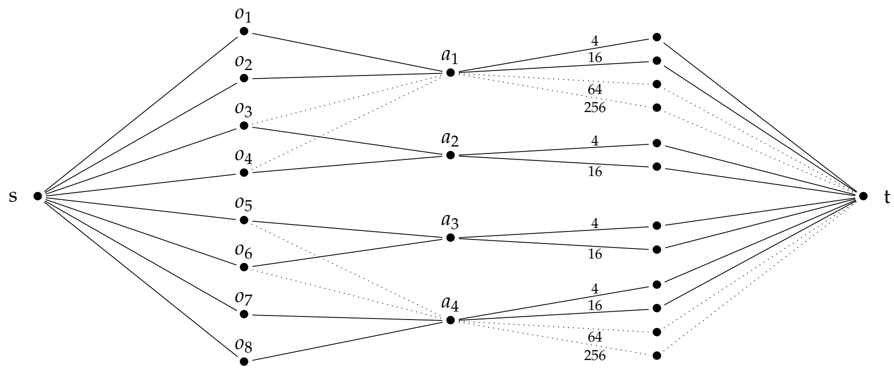

Figure 3: Flow network constructed by the Nash Flow Algorithm on the instance described in Example 3. Plain lines represent a minimum cost flow and dotted lines any other edge. The cost of the edges are indicated next to them, 0 costs have been omitted.

For the sake of contradiction, assume that π is not GEF1, that is, there exist two groups of agents $S, T \subseteq \mathcal{N}$, and a reallocation $\pi' \in \Pi(\pi_T, S)$ such that $\forall i \in S, \forall o \in \pi_i^- \cup \pi_i^+, u(\pi'_i \setminus \{o\}) \geq u(\pi_i \setminus \{o\})$ with one inequality being strict. Suppose without loss of generality that the utilities of the agents in S (resp. T), written $s_1, \ldots, s_{|S|}$ (resp. $t_1, \ldots, t_{|S|}$), are ordered increasingly. From Lemma 3 we know that for every agent $j \in T$ and $i \in S$, $u_i(\pi_j) \leq u_j(\pi_j)$. Hence, t_j is an upper bound on the utility i can receive from j.

To get a GEF1 violation, a reallocation of the items in π_T should give utility at least $s_i + 1$ to every agent $i \in S$ with one agent receiving strictly more. Let us denote by $j^* \in T$ the index of the first t_j such that $t_j > s_j + 1$. From the second Claim, it cannot be the case that $s_{j^*} < 0$. Let us then assume that $s_{j^*} \geq 0$. From our first Claim, we know that in this case j^* does not have any item considered as goods for the agent corresponding to s_{j^*}, written i^*. Since utilities are ordered increasingly, it is also the case for every agent $j > j^*$. Hence i^* can only receive goods from agents $j < j^*$. However, for every agent $j < j^*$, we have $s_j = t_j + 1$. These agents can thus not provide enough goods to all agents $i \leq i^*$ to get GEF1 violation. This contradicts the existence of such agent j^*, which in turns contradicts the existence of a GEF1 violation. We have thus proved that π is both leximin-optimal and GEF1. $\square$

From Lemma 5 and 6, we derive the statement for GEF1 allocations.

Theorem 2. *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ be an instance where $(u_i)_{i \in \mathcal{N}}$ describes a profile of ternary symmetric utility functions. A GEF1 allocation always exists and can be computed in polynomial time via the Ternary Flow Algorithm.*

Proof. Let π be the allocation returned by the Ternary Flow Algorithm. By Lemma 5, π is leximin-optimal for normalized preferences and thus GEF1 for these preferences (Lemma 6). Since no interpersonal comparison are required for GEF1, any allocation satisfying it for normalized preferences also does for non-normalized preferences. Allocation π therefore satisfies GEF1. $\square$

Let us illustrate on an example how Algorithm 2 proceeds.

Example 4. *Consider once again the instance presented in Example 3. The network flow constructed by the Nash Flow Algorithm is presented in Figure 3. The allocation corresponding to the minimum cost integer flow is $\pi = \langle \{o_1, o_2\}, \{o_3, o_4\}, \{o_5, o_6\}, \{o_7, o_8\} \rangle$. Since all items are considered as good by at least one agent, the allocation returned by the Nash Flow Algorithm is also the one returned by the Ternary Flow Algorithm.*

7 Non-existence of fair allocations

In the light of theorems 1 and 2 it may seems like leximin-optimality is closely linked to GEF1. We show in the following example that this is not true for more general preferences. Caragiannis et al. (2016) proved that for additive preferences over goods, the leximin solution does not always satisfies EF1, we provide similar example for chores.

Example 5. *Consider the following instance with four items from o_1 to o_4 and trhee agents a_1, a_2 and a_3 whose preferences are additive such that the weights for the singleton are as follows.*

	o_1	o_2	o_3	o_4
a_1	$\boxed{0}$	$-1/3$	$-1/3$	$-1/3$
a_2	$-5/7$	$\boxed{0}$	$\boxed{-1/7}$	$\boxed{-1/7}$
a_3	0	0	$-1/2$	$-1/2$

We call $\pi = \langle \{o_1\}, \{o_2, o_3, o_4\}, \emptyset \rangle$ the squared allocation. This allocation is leximin-optimal, however agent a_2 envies both agent a_1 and a_3, it is thus not EF1 hence not GEF1.

In the remainder of this section, we present negative existence results for many GEF-related concepts. We show in particular that as soon as we allow for groups of different size, existence of fair allocations cannot be guaranteed.

One of the main results presented by Conitzer et al. (2019) is that both of their relaxations of group-fairness are implied by local Nash-optimality, a relaxation Nash-optimality, when there are only goods and preferences are additive. This is particularly interesting as it proves the existence of group-fair up to one good allocations and provides a pseudo polynomial time algorithm to compute such allocations.

Definition 14 (Local Nash-optimality). *Let $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}^+$ be an instance with only goods, an allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$ is said to be locally Nash-optimal if one cannot improve the Nash welfare by simply transferring one good from one agent to another:*

$$\forall i, j \in \mathcal{N}, \forall o \in \pi_j, u_j(o) > 0 \text{ and } u_i(\pi_i) \times u_j(\pi_j) \geq u_i(\pi_i \cup \{o\}) \times u_j(\pi_j \backslash \{o\}).$$

As GEF1 is equivalent to Conitzer et al.'s relaxations when considering group of the same size, when there are only goods and preferences are additive, any locally Nash optimal allocations satisfies GEF1 for goods. We show that it is no longer true when considering groups of different size, and even more that there are no guarantee of existence of s-GEF1 allocations even for only goods and additive preferences.

Example 6. *Let us consider an instance with three agents, a_1, a_2 and a_3, and three goods, o_1, o_2 and o_3, where preferences are additive and defined as follows:*

	o_1	o_2	o_3
a_1	$\boxed{1}$	1	ϵ
a_2	1	$\boxed{1}$	ϵ
a_3	1	1	$\boxed{\epsilon}$

with $0 < \epsilon < 1/3$. We call π the allocation defined by the squared items. First, observe that $S = \{a_3\}$, $T = \{a_1, a_2, a_3\}$ and the reallocation $\pi' = \langle \{o_1, o_2, o_3\} \rangle$ are witnesses of the violation of s-GEF1 in π:

$$\frac{|S|}{|T|} u_3(\pi'_3 \backslash \{o_1\}) = \frac{1}{3} u_3(\{o_2, o_3\}) = \frac{1}{3}(1 + \epsilon) > u_3(\pi_3) = \epsilon.$$

One can see that no allocation satisfies s-GEF1 in this example. Indeed, if an agent receives more than one item then, another one would envy her up to one item. There are therefore no guarantees of existence for allocations satisfying s-GEFX and s-GEF. One can moreover see that π is not GP1, hence existence GPX and GP allocation cannot be guaranteed.

Another interesting observation is that π is Nash-optimal but is not s-GEF1.

Next, we show that while the existence of GEF1 allocations is guaranteed when there are only goods with additive preferences, it is no longer the case for monotonic preferences.

Example 7. *Let us consider two agents, a_1 and a_2, whose preferences, u_1 and u_2, are presented below. The preferences only depend on the number of items received by each agent. Agent a_2 gets positive utility only if she receives at least 3 items.*

$X \subseteq \mathcal{O}$	$u_1(X)$	$u_2(X)$
$\|X\| = 4$	10	10
$\|X\| = 3$	6	6
$\|X\| = 2$	4	0
$\|X\| = 1$	1	0
$\|X\| = 0$	0	0

In such instance, the only allocations satisfying GEF1 for $S = T = \mathcal{N}$ are the ones in which a_1 gets either all the goods, none or exactly one. None of these allocations are EF1, hence no allocation is GEF1.

One can wonder why is the case with only chores not symmetric as the one with only goods. In a very basic example, we show that maximizing of minimizing the Nash welfare could lead to an allocation that is not EF1 when there are only chores.

Example 8. *Let us consider an instance with two agents a_1 and a_2 and four chores o_1, o_2, o_3 and o_4 where preferences are additive and defined as follow:*

	o_1	o_2	o_3	o_4
a_1	-1	-100	-100	-100
a_2	-6	-2	-2	-2

The allocation maximizing the absolute value of the Nash welfare is $\langle \{o_2, o_3, o_4\}, \{o_1\} \rangle$ in which a_1 envies a_2 up to one chore. Let us now consider the following instance:

	o_1	o_2	o_3	o_4
a_1	-1	-100	-100	-100
a_2	-1	-1	-1	-1

Here, minimizing the Nash welfare leads to $\langle \{o_1\}, \{o_2, o_3, o_4\} \rangle$ where a_2 envies a_1 up to one chore.

8 Testing GEF1 is coNP-complete

We prove in this section that testing GEF1 is coNP-complete when there are only goods, only chores and both of them. The decision problem is the following.

IS-GEF1	
Instance:	An instance $I = \langle \mathcal{N}, \mathcal{O}, (u_i)_{i \in \mathcal{N}} \rangle \in \mathcal{I}$ and $\pi \in \Pi(\mathcal{O}, \mathcal{N})$.
Question:	Does π satisfy GEF1?

We use IS-GEF1$^+$ and IS-GEF1$^-$ to refer to the same decision problem when there are respectively only goods ($I \in \mathcal{I}^+$) and only chores ($I \in \mathcal{I}^-$).

Theorem 3. *The problems* IS-GEF1, IS-GEF1$^+$ *and* IS-GEF1$^-$ *are strongly coNP-complete.*

Proof for IS-GEF1$^-$. We present the reduction for the IS-GEF1$^-$ problem. By reducing the 3-PARTITION problem (Garey and Johnson, 1975), we show that checking if π violates GEF1 when there are only chores is strongly NP-complete.

3-PARTITION	
Instance:	A multi-set of $3m$ numbers $X = \{x_1, \ldots, x_{3m}\}$ such that: $\forall x \in X, 1/4 < x < 1/2$ and $\sum_{x \in X} x = m$.
Question:	Is there a partition $(X_i)_{i \in [\![1,m]\!]}$ of X such that $\forall i, \sum_{x \in X_i} = 1$?

Let $X = \{x_1, \ldots, x_{3m}\}$ be an instance of the 3-PARTITION problem. We present in the following its corresponding instance (I, π) of the IS-GEF1$^-$ problem. The set of chores is $\mathcal{O} = \{g_1, \ldots, g_m\} \cup \{h_1, \ldots, h_m\} \cup \{l_1, \ldots, l_{3m}\} \cup \{o_1, \ldots, o_{2m}\}$ and the set of agents $\mathcal{N} = \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}$. The utilities of the singletons are as follows.

	g_1	$\ldots$	g_m	h_1	$\ldots$	h_m	l_1	l_2	l_3	$\ldots$	l_{3m-2}	l_{3m-1}	l_{3m}	o_1	$\ldots$	o_m	o_{m+1}	$\ldots$	o_{2m}
a_1	$\boxed{-m-\epsilon}$	$-M$	$-M$	$\boxed{-1-\epsilon}$	$-M$	0	$-x_1$	$-x_2$	$-x_3$	$\cdots$	$-x_{3m-2}$	$-x_{3m-1}$	$-x_{3m}$	$\boxed{0}$	$-M$	$-M$	$-M$	$-M$	$-M$
$\vdots$	$-M$	$\ddots$	$-M$	$-M$	$\ddots$	$-M$	$-x_1$	$-x_2$	$-x_3$	$\cdots$	$-x_{3m-2}$	$-x_{3m-1}$	$-x_{3m}$	$-M$	$\ddots$	$-M$	$-M$	$-M$	$-M$
a_m	$-M$	$-M$	$\boxed{-m-\epsilon}$	$-M$	$-M$	$\boxed{-1-\epsilon}$	$-x_1$	$-x_2$	$-x_3$	$\cdots$	$-x_{3m-2}$	$-x_{3m-1}$	$-x_{3m}$	$-M$	$-M$	$\boxed{0}$	$-M$	$-M$	$-M$
b_1	$-x_2-x_3$	$-M$	$-M$	$-M$	$-M$	$-M$	$\boxed{-x_1}$	$\boxed{-x_2}$	$\boxed{-x_3}$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$\boxed{0}$	$-M$	$-M$
$\vdots$	$-M$	$\ddots$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$\ddots$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$\ddots$	$-M$
b_m	$-M$	$-M$	$-x_{3m-1}-x_{3m}$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$-M$	$\boxed{-x_{3m-2}}$	$\boxed{-x_{3m-1}}$	$\boxed{-x_{3m}}$	$-M$	$-M$	$-M$	$-M$	$-M$	$\boxed{0}$

where $\epsilon > 0$ is a constant small enough, M is a constant greater than $m+1$ and the x_i are assumed to be ordered in a decreasing order: $\forall i \in [\![1,3m]\!], x_i \geq x_{i+1}$.

Let us explain these preferences in detail. Agent a_i, $i \in [\![1,m]\!]$, gives value $-m-\epsilon$ to chore g_i and $-M$ ot any other "g" chore. Moreover, she values $-1-\epsilon$ chore h_i, 0 chore h_{i-1} (chore h_m for agent a_1) and $-M$ the other "h" chores. Her valuation for "l" chores follows the opposite of the values in X. Finally, her utility for "o" chores is 0 for o_i and $-M$ for the others.

Agent $b_i, i \in [\![1,m]\!]$, values $-x_{3i-1} - x_{3_i}$ chores g_i, and $-M$ all other "g" chores. All "h" chores give her $-M$ utility. Chores l_{3i-2}, l_{3i-1} and l_{3i} respectively provide her x_{3i-2}, x_{3i-1} and x_{3i} utility while she values $-M$ any other "l" chore. Finally her utility for "o" chores is 0 for o_{i+m} and $-M$ for the others.

The initial allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$, given as a entry of the IS-GEF1$^-$ problem and represented by the boxed items in the previous table, is defined as follow:

$$\pi_{a_i} = \{g_i\} \cup \{h_i\} \cup \{o_i\}, \forall i \in [\![1,m]\!],$$
$$\pi_{b_i} = \{l_{3i-2}, l_{3i-1}, l_{3i}\} \cup \{o_{m+i}\}, \forall i \in [\![1,m]\!].$$

Next, we prove that allocation π violates GEF1 if and only if there exists a partition $(X_i)_{i \in [\![1,m]\!]}$ of X satisfying the 3-PARTITION conditions.

We first show that if the allocation π is not GEF1, then the two groups S and T witnessing this violation can only be $\mathcal{N}$, that is $S = T = \mathcal{N}$. To do so, let us consider any two groups S and T and a reallocation $\pi' \in \Pi(\pi_T, S)$ such that:

$$\forall a \in S, \forall o \in \pi_a, u_a(\pi'_a) \geq u_a(\pi_a \backslash \{o\}), \tag{7}$$

with at least one inequality being strict. Such S, T and π' define a violation of GEF1. Note that as (7) should hold for any item $o \in \pi_a$, it should hold in particular if o is the worst chore in π_a that is g_i for agents a_i and l_{3i-2} for agent b_i (remember that we assumed x_i are decreasingly ordered). Hence (7) can be reformulated as:

$$\forall a \in S, \begin{cases} u_a(\pi'_a) \geq -1-\epsilon & \text{if } a \in \{a_1, \ldots, a_m\}, \\ u_a(\pi'_a) \geq -x_{3i-1} - x_{3i} & \text{if } a = b_i, i \in [\![1,m]\!], \end{cases} \tag{8}$$

with one inequality being strict.

Let us show that (8) can be satisfied if and only if $S = T = \mathcal{N}$. To do so, we first prove that $S = T$. Then, we show that the number of "a" agents in T is equal to the number of "b" agents in T. Finally we prve that if agent a_i (resp. a_1) is in T, then agent a_{i-1} (resp. a_m) should also be in T, proving that $\{a_1, \ldots, a_m\} \in T$. All these facts put together show the claim.

Consider agent $c \in T$ that can either be an "a" or "b" agent. Let k be the index such that $k = i$ if $c = a_i$ and $k = i + m$ if $c = b_i$. Observe that chore o_k is in π_c and it can only be allocated to agent c for (8) to be satisfied as every other agent have utility $-M$ for it. We have thus $T \subseteq S$, and as $|S| = |T|$ we have $S = T$.

Let us then introduce two notations, n_a and n_b, respectively corresponding to the number of "a" agents and "b" agent in T, that is $n_a = |\{a_1, \ldots, a_m\} \cap T|$ and $n_b = |T| - n_a$. Note that if $a_i \in T$ then b_i should be in S as $g_i \in \pi_{a_i}$ can only be allocated to b_i to satisfy (8). Remember that $S = T$, hence $b_i \in T$ and thus $n_a \leq n_b$.

Moreover, observe that $T \subseteq \{b_1, \ldots, b_m\}$ would violated (8) as "b" agents are interested only in the "l" chores they own among all "l" chores. Hence, no reallocation among only "b" agents can be improving. Let us then assume that there exists agent a_i in T. As already mentioned, we have $g_i \in \pi'_{b_i}$, hence $\pi'_{b_i} = \{g_i, o_{i+m}\}$ and (8) for b_i is at equality. All "l" chores in π_T should then be reallocated to "a" agents as "b" can not received any additional chore. Remember that $\forall x \in X, 1/4 < x < 1/2$, hence any subset of X of size strictly greater than 4 has a sum strictly greater than 1. For a suitable ϵ it is then impossible to reallocate more than 3 "l" items to an "a" agents without its utility being below $-1 - \epsilon$. This implies that $n_b \leq n_a$ hence $n_a = n_b$.

In addition, there are exactly $3n_b$ "l" chores in π_T and as $n_a = n_b$, each "a" agent should be reallocated exactly 3 "l" items. It is then not possible for agent a_i to receive chore h_i, this chore should then be allocated to agent a_{i-1} (a_m for $a_i = a_1$). Hence, if $a_i \in T$ we should have $a_{i-1} \in S$ (a_m for $a_i = a_1$) and thus in T as $S = T$. This yields that all "a" agents are in T and thus also all "b" agents as $n_a = n_b$. We have thus $S = T = \mathcal{N}$.

We now prove that for $S = T = \mathcal{N}$, there exist a reallocation π' satisfying (8) if and only if there exists a partition $(X_i)_{i \in [\![1,m]\!]}$ of X satisfying the conditions of the 3-PARTITION problem.

Note that each g_i chore should be allocated to b_i in π', hence all "l" chores should be divided among "a" agents. "h" chores are allocated to "a" agents receiving 0 utility for it, similar reallocation is done for "o" chores. Hence (8) is satisfied if and only if it is possible to divide the "l" items into m parts of sum smaller than $1 + \epsilon$, that is of sum 1. For a suitable ϵ this is equivalent to the existence of a partition of X satisfying the conditions of the 3-PARTITION problem.

Finally this reduction is clearly done in polynomial-time which concludes the proof. $\square$

Sketch of the proof for IS-GEF1$^+$. Let us consider $X = \{x_1, \ldots, x_{3m}\}$ an instance of the 3-PARTITION problem. We construct an instance of the IS-GEF1$^+$ problem as follows. The set of agents is $\mathcal{N} = \{a_1, \ldots, a_m, b\}$, the set of items is $\mathcal{O} = \{g_1, \ldots, g_m\} \cup \{h_i^j \mid i, j \in [\![1,m]\!], i \neq j\} \cup \{l_1, \ldots, l_{3m}\} \cup \{l_1^*, l_2^*\}$ and the preferences of the agent are given in the following table:

	g_1	$\ldots$	g_m	h_1^2	$\ldots$	h_1^m	$\ldots$	h_m^1	$\ldots$	h_m^{m-1}	l_1	$\ldots$	l_{3m}	l_1^*	l_2^*
a_1	$\boxed{m+1-\epsilon}$	0	0	$\frac{m}{m-1}$	$\frac{m}{m-1}$	$\frac{m}{m-1}$	0	$\boxed{0}$	0	0	x_1	$\cdots$	x_{3m}	$\boxed{0}$	0
$\vdots$	0	$\ddots$	0	0	0	0	$\ddots$	0	0	0	x_1	$\cdots$	x_{3m}	0	0
a_m	0	0	$\boxed{m+1-\epsilon}$	0	0	$\boxed{0}$	0	$\frac{m}{m-1}$	$\frac{m}{m-1}$	$\frac{m}{m-1}$	x_1	$\cdots$	x_{3m}	0	$\boxed{0}$
b	0	0	0	0	0	0	0	0	0	0	$\boxed{x_1}$	$\boxed{\cdots}$	$\boxed{x_{3m}}$	m	m

where $\epsilon > 0$ is a constant small enough. The initial allocation $\pi \in \Pi(\mathcal{O}, \mathcal{N})$, given as a entry of the IS-GEF1$^+$ problem, is presented by the boxed items in the previous table. It is defined as follow:

$$\begin{aligned}
\pi_{a_1} &= \{g_1\} \cup \{h_i^1 \mid j \in [\![2,m]\!]\} \cup \{l_1^*\}, \\
\pi_{a_m} &= \{g_m\} \cup \{h_j^m \mid j \in [\![1,m-1]\!]\} \cup \{l_2^*\}, \\
\pi_{a_i} &= \{g_i\} \cup \{h_j^i \mid j \in [\![1,m]\!], j \neq i\}, \forall i \in [\![2,m-1]\!], \\
\pi_b &= \{l_1, \ldots, l_{3m}\}.
\end{aligned}$$

We claim that allocation π violates GEF1 if and only if there exists a partition $(X_i)_{i \in [\![1,m]\!]}$ of X satisfying the 3-PARTITION conditions. $\square$

Finally, since any instance of the IS-GEF1$^+$ or IS-GEF1$^-$ problems is also an instance of the IS-GEF1 problem, the previous proof also prove that IS-GEF1 is coNP-complete.

In this proof, the only possible violations of GEF1 are such that $S = T = \mathcal{N}$. Hence, checking whether an allocation satisfies the Pareto-optimality relaxation derived from GEF1, is also coNP-complete. These results are similar in flavour to the results of de Keijzer, Bouveret, Klos, and Zhang (2009) and Aziz, Biro, Lang, Lesca, and Monnot. (2019a) that testing Pareto-optimality is coNP-complete.

9 Conclusions

Inspired by the group envy-freeness concept and its relaxations for the case of divisible goods, we formalized several relaxations of the concept for indivisible goods and chores. The concepts have both fairness and efficiency flavours. Our definitions are general and our key concepts work well for ordinal preferences as well as cardinal utilities involving goods and chores. We clarified the relation of GEF1 with other fairness concepts and presented several positive and negative computational results.

Several interesting questions arise as a result of our study. The main question left open is the existence of GEF1 allocations when there are goods and chores. The question has been answered positively in the case of goods. However the proof involves the Nash social welfare which cannot be used with chores. Considering that protection of groups is one of the central concerns in new research on algorithmic fairness, we envisage GEF1 and its variants to spur further interesting work in the area.

References

M. Aleksandrov and T. Walsh. Group envy freeness and group Pareto efficiency in fair division with indivisible items. In *Proceedings of the 41st KI Conference*, 2018.

H. Aziz and S. Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. In *Proceedings of the 57th Symposium on Foundations of Computer Science (FOCS)*, 2016.

H. Aziz, G. Rauchecker, G. Schryen, and T. Walsh. Algorithms for max-min share fair allocation of indivisible chores. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, 2017.

H. Aziz, P. Biro, J. Lang, J. Lesca, and J. Monnot. Efficient reallocation under additive and ordinal preferences. *Theoretical Computer Science*, 2019a.

H. Aziz, I. Caragiannis, A. Igarashi, and T. Walsh. Fair allocation of combinations of indivisible goods and chores. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, 2019b.

S. Barman and S. K. Krishnamurthy. Approximation algorithms for maximin fair division. In *Proceedings of the 18th ACM Conference on Electronic Commerce (ACM-EC)*, 2017.

N. Benabbou, M. Chakraborty, E. Elkind, and Y. Zick. Fairness towards groups of agents in the allocation of indivisible items. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, 2019.

M. Berliant, W. Thomson, and K. Dunz. On the fair division of a heterogeneous commodity. *Journal of Mathematical Economics*, 21(3):201–216, 1992.

A. Bogomolnaia, H. Moulin, and R. Stong. Collective choice under dichotomous preferences. *Journal of Economic Theory*, 122(2):165–184, 2005.

A. Bogomolnaia, H. Moulin, F. Sandomirskiy, and E. Yanovskaya. Dividing goods or bads under additive utilities. *arXiv preprint arXiv:1608.01540*, 2016.

A. Bogomolnaia, H. Moulin, F. Sandomirskiy, and E. Yanovskaya. Competitive division of a mixed manna. *Econometrica*, 85(6):1847–1871, 2017.

S. Bouveret, Y. Chevaleyre, and N. Maudet. Fair allocation of indivisible goods. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia, editors, *Handbook of Computational Social Choice*, pages 284–310. Cambridge University Press, 2016.

S. J. Brams and P. C. Fishburn. Approval voting. *American Political Science Review*, 72(3):831–847, 1978.

S. J. Brams and A. D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.

E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, and M. Kyropoulou. The efficiency of fair division. *Theory of Computing Systems*, 50(4):589–610, 2012.

I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah, and J. Wang. The unreasonable fairness of maximum nash welfare. In *Proceedings of the 17th ACM Conference on Electronic Commerce (ACM-EC)*, 2016.

V. Conitzer, R. Freeman, N. Shah, and J. W. Vaughan. Group fairness for the allocation of indivisible goods. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI)*, 2019.

A. Darmann and J. Schauer. Maximizing nash product social welfare in allocating indivisible goods. *European Journal of Operational Research*, 247(2):548–559, 2015.

B. de Keijzer, S. Bouveret, T. Klos, and Y. Zhang. On the complexity of efficiency and envy-freeness in fair division of indivisible goods with additive preferences. In *Proceedings of the 1st ADT Conference*, 2009.

S. Dehghani, A. Farhadi, M. HajiAghayi, and H. Yami. Envy-free chore division for an arbitrary number of agents. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2018.

E. Eisenberg. Aggregation of utility functions. *Management Science*, 7(4):337–350, 1961.

B. Fain, K. Munagala, and N. Shah. Fair allocation of indivisible public goods. In *Proceedings of the 19th ACM Conference on Electronic Commerce (ACM-EC)*, 2018.

D. S. Felsenthal. On combining approval with disapproval voting. *Behavioral Science*, 34(1):53–60, 1989.

D. K. Foley. Resource allocation and the public sector. In *Yale Economics Essays*, volume 7. 1967.

D. Gale. *The theory of linear economic models*. McGraw Hill, 1960.

M. Gardner. *Aha! Aha! insight*. Scientific American, 1978.

M. R. Garey and D. S. Johnson. Complexity results for multiprocessor scheduling under resource constraints. *SIAM Journal on Computing*, 4(4):397–411, 1975.

E. Halevi and S. Nitzan. Envy-free cake-cutting among families. *arXiv preprint arXiv:1607.01517*, 2016.

P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 1(1):26–30, 1935.

F. Husseinov. A theory of a heterogeneous divisible commodity exchange economy. *Journal of Mathematical Economics*, 47(1):54–59, 2011.

M. Kyropoulou, W. Suksompong, and A. A. Voudouris. Almost envy-freeness in group resource allocation. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, 2019.

J. Lang and J. Rothe. Fair division of indivisible goods. In J. Rothe, editor, *Economics and Computation*, pages 493–550. 2016.

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (ACM-EC)*, 2004.

P. Manurangsi and W. Suksompong. Asymptotic existence of fair divisions for groups. *Mathematical Social Sciences*, 89:100–108, 2017.

H. Moulin. *Fair division and collective welfare*. MIT press, 2004.

H. Moulin. Fair division in the age of internet. 2018.

A. Othman and T. Sandholm. Envy quotes and the iterated core-selecting combinatorial auction. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI)*.

E. Peterson and F. E. Su. Four-person envy-free chore division. *Mathematics Magazine*, 75(2):117–122, 2002.

B. Plaut and T. Roughgarden. Almost envy-freeness with general valuations. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2018.

A.E. Roth and M. Sotomayor. Two-sided matching. In R.J. Aumann and S. Hart, editors, *Handbook of game theory with economic applications*, chapter 16. Elsevier, 1992.

D. Schmeidler and K. Vind. Fair net trades. *Econometrica*, 40(4):637, 1972.

A. Schrijver. *Combinatorial optimization: polyhedra and efficiency*. Springer, 2003.

E. Segal-Halevi. Fairly dividing a cake after some parts were burnt in the oven. In *Proceedings of the 17th International Joint Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, 2018.

E. Segal-Halevi and W. Suksompong. Democratic fair allocation of indivisible goods. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI)*, 2018.

H. Steinhaus. The problem of fair division. *Econometrica*, 16:101–104, 1948.

H. Steinhaus. Sur la division pragmatique. *Econometrica*, pages 315–319, 1949.

F. E. Su. Rental harmony: Sperner's lemma in fair division. *The American mathematical monthly*, 106 (10):930–942, 1999.

L.-G. Svensson. Large indivisibles: an analysis with respect to price equilibrium and fairness. *Econometrica*, pages 939–954, 1983.

T. Todo, R. Li, X. Hu, T. Mouri, A. Iwasaki, and M. Yokoo. Generalizing envy-freeness toward group of agents. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, 2011.

K. Vind. Lecture notes for economics, 1971.

H. P. Young. *Equity: in theory and practice*. Princeton University Press, 1995.

L. Zhou. Strictly fair allocations in large exchange economies. *Journal of Economic Theory*, 57(1):158–175, 1992.