Fairness in Multiwinner Elections

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Lecture Two

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Plan

Lecture Plan:

- ⋆ Cohesive groups.
- ⋆ Justified Representation.
- ⋆ PAV again.
- ⋆ The core.

Approval-based Multiwinner Election Model

We focus on the general approval-based model today.

- ⋆ A set of candidates $C = \{a, b, c, \dots\}$ of size $m \geqslant 2$.
- ⋆ A set of agents $N = \{1, \dots, n\}$ of size n.
- ⋆ Each agent $i \in N$ submits an *approval ballot* $A_i \subseteq C$.
- ⋆ A vector of these n approval ballots is called an *approval profile* $\boldsymbol{A} = (A_1, \dots, A_n)$.
- ⋆ Take $\mathcal{P}_k(C)$ as the set of all committees W of size k (all feasible outcomes).
- ⋆ An election instance is denoted by the pair $(\boldsymbol{A}, k)$.
- ⋆ An approval-based multiwinner voting rule $\mathcal{R}$ maps election instances to a non-empty set $\mathcal{R}(\boldsymbol{A}, k)$ of k-size committees from $\mathcal{P}_k(C)$ with elements of $\mathcal{R}(\boldsymbol{A}, k)$ called *winning committees* and denoted as W. We deal with *irresolute* rules.

w-Thiele Methods

Recall the class of w-Thiele methods.

Definition (w-Thiele Method)

Given a profile $\boldsymbol{A} = (A_1, \dots, A_n)$, the following is the *$w$-Thiele method* that is induced by some weight function $w : \mathbb{N}_{>0} \to \mathbb{R}^+$:

$$\operatorname*{argmax}_{W \in \mathcal{P}_k(C)} \sum_{i \in N} \sum_{j=1}^{|W \cap A_i|} w(j)$$

- ⋆ AV with $w_{AV}(j) = 1$ for all $j \in \mathbb{N}_{>0}$.
- ⋆ α-CC with $w_{CC}(j) = 1$ if $j = 1$, and $w_{CC}(j) = 0$ otherwise.
- ⋆ PAV with $w_{PAV}(j) = 1/j$ for all $j \in \mathbb{N}_{>0}$.

Cohesiveness

Cohesive Groups

A central notion we'll see today is that of *cohesive groups*. Consider an election instance $(\boldsymbol{A}, k)$ with agent set N.

Definition (ℓ-cohesive group)

For $\ell \in \{1, \ldots, k\}$, we say a group of agents $N^* \subseteq N$ is ℓ-cohesive if $|N^*| \geqslant \ell \cdot n/k$, and $|\bigcap_{i \in N^*} A_i| \geqslant \ell$.

If the entire agent population determines k committee positions, then a group of agents that represents an ℓ/k-th fraction of the population is able to control $k \cdot \ell/k = \ell$ spots in the committee. How do we define this 'control' of ℓ committee spots?

Cohesive Groups

For some ℓ-cohesive group N^*, every candidate in $\bigcap_{i \in N^*} A_i$ is a representative for every agent in N^*. So, for an axiom, we might require that every agent in an ℓ-cohesive group get at least ℓ representatives in the winning committee W.

Actually, no rule can satisfy this, even when $\ell = 1$.

Example: Take four alternatives $\{a, b, c, d\}$ when $k = 3$. Of 9 total agents, 2 agents approve of a and 2 agents approve of d, while 1 agent each approves b, c, $\{a, b\}$, $\{b, c\}$, and $\{c, d\}$. So every alternative is supported by a cohesive group of size $3 \geqslant 9/3$, but we cannot elect every alternative.

Justified Representation

Justified Representation

Now we use cohesive groups to define a proportionality axiom.

Definition (Justified Representation)

A committee $W \subseteq C$ of size k satisfies *Justified Representation (JR)* if for every 1-cohesive group $N^* \subseteq N$ (so, $|N^*| \geqslant n/k$ and $|\cap_{i \in N^*} A_i| \geqslant 1$), there exists some agent $i \in N^*$ such that $A_i \cap W \neq \emptyset$.

A rule $\mathcal{R}$ satisfies JR, if for every $(\boldsymbol{A}, k)$, each winning committee $W \in \mathcal{R}(\boldsymbol{A}, k)$ satisfies JR.

Q: Does this seem like a reasonable requirement?

AV and JR

When $k \geqslant 3$, the AV rules fails JR.

Example: Let $k \geqslant 3$ and take a set of candidates $C = \{c_0, c_1, \ldots, c_k\}$. Consider a set of agents such that $n = k$, where agent 1 only approves of candidate c_0, and every other agent $j \in \{2, \ldots, k\}$ approves of every all candidates in $\{c_1, \ldots, c_k\}$. Satisfying JR requires the selection of c_0, however, AV will select $\{c_1, \ldots, c_k\}$.

We now ask, does PAV satisfy it?

PAV and JR

Theorem (Aziz et al., 2017)

PAV satisfies JR.

Proof.

- ⋆ Take an election instance $(\boldsymbol{A}, k)$ and let $s = {}^{n}/_{k}$. Let W be the committee returned by PAV for $(\boldsymbol{A}, k)$.
- ⋆ Assume that there exists a ℓ-cohesive group $N' \subseteq N$ such that $\bigcap_{i \in N^*} A_i \neq \emptyset$, but, it holds that $W \cap \bigcup_{i \in N^*} A_i = \emptyset$. Take some $c \in \bigcap_{i \in N^*} A_i$, i.e., all agents in N^* approve of c.
- ⋆ Work towards a contradiction by showing that c should've been part of the winning committee.
- ⋆ The *marginal contribution* of some candidate $w \in W$ is the difference between the scores of W and $W \setminus \{w\}$. The sum of marginal contributions is $m(W)$.

PAV and JR

Proof.

- ⋆ If we add c to the winning committee, this would increase the PAV score of W by at least s (i.e., agents in the cohesive group are satisfied).
- ⋆ So if we show that some candidates $w \in W$ has a marginal contribution to W that is strictly less than s, then we could just swap c in for this w, and N^* would be satisfied.
- ⋆ We aim to show that $m(W) \leqslant s(k-1)$, then, by the pigeonhole principle, we have proved the claim.
- ⋆ Take the set of remaining agents $N \setminus N^*$. We know that $n \leqslant s \cdot k$, so it holds that $|N \setminus N^*| \leqslant n - s \leqslant s(k-1)$.

PAV and JR

Proof.

- ⋆ Now, let's choose one of the agents $i \in N \setminus N^*$, and set $j = |A_i \cap W|$.
- ⋆ If $j > 0$, this chosen agent $i \in N \setminus N^*$ contributes a $1/j$ to the total marginal contribution of each candidate in $A_i \cap W$. In terms of the sum of marginal contributions $m(W)$, agent i's contribution is exactly 1.
- ⋆ And if $j = 0$? Then this agent i does not contribute to $m(W)$.
- ⋆ Now, it holds that $m(W) \leqslant |N \setminus N^*| \leqslant s(k-1)$.
- ⋆ So, PAV would've elected a candidate $c \in \bigcap_{i \in N^*} A_i$ that satisfies the 1-cohesive group N^* which we assumed to be underrepresented.

□

This is good news! One may wonder, from the connection to PAV in the apportionment setting, whether the sequential PAV rule satisfies JR?

Sequential PAV

Recall that sequential PAV works in k rounds and returns the committee W_k.

Starting with an empty committee $W_0 = \emptyset$, in every round r, the rule adds a candidate $c \in C$, to set $W_r = W_{r-1} \cup \{c\}$, that maximises (and use tiebreaking, if necessary):

$$\sum_{i \in N} \sum_{j=1}^{|A_i \cap W_{r-1} \cup \{c\}|} \frac{1}{j}$$

Unfortunately, sequential PAV fails JR in the general case.

Theorem (Sánchez-Fernández et al., 2017)

Sequential PAV satisfies JR for $k \leqslant 5$, and fails JR for $k \geqslant 6$.

The failure for $k = 6$ involves a counterexample with **5992 candidates**.

Extended Justified Representation

The next example shows a slight 'issue' with JR.

Example: Take a set of candidates $C = \{a, b, c, d\}$ with $k = 3$. Let's say 100 agents participate with one agent only approving of a, one agent only approving of b and then remaining 98 agents approving of both c and d. Consider the committee $\{a, b, c\}$. This committee provides JR.

It is clear that JR is not 'concerned' if some very large group of agent get more than one representative. So, to address this issue for larger groups, we look at a strengthening of JR.

Extended Justified Representation

Definition (Extended Justified Representation)

A committee $W \subseteq C$ of size k satisfies *Extended Justified Representation (EJR)* if for every $\ell \in \{1, \ldots, k\}$ and every ℓ-cohesive group $N^* \subseteq N$, then there exists some $i \in N^*$ such that $|A_i \cap W| \geqslant \ell$.

When $\ell = 1$, this is just JR. Intuitively, EJR scales up JR's requirements for the size of the cohesive group, and the amount of representatives that group deserves, by ℓ.

Theorem (Aziz et al., 2017)

PAV satisfies EJR.

Next, we show that PAV satisfies EJR.

PAV and EJR

Proof.

- ⋆ Assume that there exists a ℓ-cohesive group $N^* \subseteq N$ such that $\bigcap_{i \in N^*} A_i \geqslant \ell$, but, it holds that $|W \cap \bigcup_{i \in N^*} A_i| < \ell$. Take $|N^*| = s \geqslant \ell n/k$.
- ⋆ Take a candidate c to be the candidate in $\bigcap_{i \in N^*} A_i$ that was not elected.
- ⋆ The agents in N^* have, at the very most, $\ell - 1$ candidates in W that represent them. So, the marginal contribution of adding c is at least $s \cdot 1/\ell \geqslant n/k$.
- ⋆ Now, we aim to show that sum of marginal contributions $m(W)$ is at most n.
- ⋆ Take some $w \in W$ with the smallest marginal contribution, this is at most n/k.
- ⋆ If this marginal contribution is strictly less than n/k, we could just swap c in for w and this would improve the PAV score of W.
- ⋆ Now, say w's marginal contribution is equal to n/k (the same holds for all other members of W).

PAV and EJR

Proof.

- ⋆ Well, we know that PAV satisfies JR, so for some agent in the ℓ-cohesive group N^*, it holds that $W \cap A_i \neq \emptyset$. Let's take a candidate $w' \in W \cap A_i$.
- ⋆ Set a new committee $W' = (W \setminus \{w'\}) \cup \{c\}$.
- ⋆ Note that the agent $i \in N^*$ that approves of w', approves at most $\ell - 2$ candidates in $W \setminus \{w'\}$, so adding c to the committee $W \setminus \{w'\}$, adds at least $1/\ell-1$ satisfaction to agent i.
- ⋆ Therefore, removing w' and then adding c, increases the PAV score by at least $(s-1) \cdot 1/\ell + 1/\ell-1$ which is strictly more than n/k.
- ⋆ So, the PAV score of W' is more than W's PAV score, which is a contradiction.

$\square$

In fact, PAV is the only w-Thiele method that satisfies EJR (α-CC fails it). So this axiom characterises PAV and lends further support behind using PAV.

Perfect Representation

Let's take a look at the following representation axiom motivated by fairness in parliamentary elections.

Definition (Perfect Representation)

Consider a ballot profile $\boldsymbol{A} = (A_1, \ldots, A_n)$ over a set of candidates C, and a target committee size $k \leqslant |C|$, such that k divides n. We say that a committee W with $|W| = k$, provides *perfect representation (PR)* for $(\boldsymbol{A}, k)$ if it is possible to partition N into k pairwise disjoint subsets $N_1, \ldots, N_k$, each of size n/k, and assign a distinct candidate from W to each of these subsets so that it holds for each $\ell \in \{1, \ldots, k\}$, that all agents in N_ℓ approve of their assigned member of W.

A rule $\mathcal{R}$ satisfies PR, if for every $(\boldsymbol{A}, k)$, each winning committee $W \in \mathcal{R}(\boldsymbol{A}, k)$ satisfies PR, whenever such a committee exists.

Perfect Representation

This axiom seems fairly weak axiom (and only makes sense when k divides n), but, we show that it is incompatible with EJR.

Theorem (Sánchez-Fernández et al., 2017)

There exists an approval profile $\boldsymbol{A}$ and a committee size k such that the set of committees that satisfy PR for $(\boldsymbol{A}, k)$ is non-empty, but does not provide EJR.

Proof.

Take a set of candidates $C = \{c_1, \ldots, c_6\}$, and an approval profile $\boldsymbol{A} = (A_1, \ldots, A_8)$ such that $i = \{1, 2, 3, 4\}$ where $A_i = \{c_i\}$ and $A_{i+4} = \{c_i, c_5, c_6\}$. A committee $\{c_1, c_2, c_3, c_4\}$ is the unique committee for election instance $(\boldsymbol{A}, 4)$. To see that this committee fails EJR, note that agents who submit the approval ballots A_5, A_6, A_7 and A_8, form a 2-cohesive group of agents, but, they all approve of only 1 committee member. $\square$

So, is it possible to take cohesive groups into account while satisfying PR?

Proportional Justified Representation

We can define a weakening of EJR that is compatible with PR.

Definition (Proportional Justified Representation)

A committee $W \subseteq C$ of size k satisfies *Proportional Justified Representation (PJR)* if for every $\ell \in \{1, \dots, k\}$ and every ℓ-cohesive group $N^* \subseteq N$, then $|W \cap \bigcup_{i \in N^*} A_i| \geqslant \ell$ must hold.

As EJR is stronger than PJR, PAV satisfies this PJR axiom.

Theorem (Sánchez-Fernández et al., 2017)

For every $(\boldsymbol{A}, k)$, if a committee W of size k provides PR, then it also provides PJR.

PR and PJR

Proof.

- ⋆ As there is a committee W that satisfies PR, then it holds that k divides n. Let's set $W = \{w_1, \ldots, w_k\}$.
- ⋆ Since W provides PR, there exists k pairwise disjoint subsets $N_1, \ldots, N_k$ of size $^n/_k$ each such that all agents in N_i approve of w_i for each $i \in \{1, \ldots, k\}$.
- ⋆ Take a set of agents $N^* \subseteq N$ and a positive integer ℓ such that $|N^*| \geqslant \ell \cdot {^n/_k}$. By the pigeonhole principle, N^* has a nonempty intersection with at least ℓ of the sets $N_1, \ldots, N_k$.
- ⋆ Now, since every agent in the intersection $N^* \cap N_i$ approves of w_i, it follows that the number of candidates in W approved by some agent in N^* must be at least ℓ.

□

The picture looks quite good for PAV thus far, so we ask, what are its limits?

The Core

The Core

The following is adapted from the core stability notion of cooperative game theory.

Definition (The Core)

Given an election instance $(\boldsymbol{A}, k)$, we say that a committee W is in the core if for each non-empty $N^* \subseteq N$ and $C' \subseteq C$ with

$$\frac{|C'|}{k} \leqslant \frac{|N^*|}{n}$$

there exists an agent $i \in N^*$ such that $| A_i \cap C' | \leqslant | A_i \cap W |$. If a rule $\mathcal{R}$ always returns winning committees that are in the core for every election instance $(\boldsymbol{A}, k)$, then we say that this rule satisfies the core property.

So, if a group of agents N^*, that are $\alpha\%$ of the population, can find a subset of the candidates of size $\alpha\% \cdot k$ such that every member of N^* approves of more candidates in C' than the committee W, then they would ‘deviate’ from W.

The Core and EJR

Let's see that the core property really is stronger than EJR.

Assume that a rule $\mathcal{R}$ satisfies the core property and consider an instance $(\boldsymbol{A}, k)$, a winning committee W, and an ℓ-cohesive group of agents N^*. Let C' be the set of ℓ candidates that are approved by all the agents in N^* (such candidates exist because N^* is ℓ-cohesive). Since W is in the core, there must exist an agent $i \in N^*$ such that $|A_i \cap W| \geqslant |A_i \cap C'| = \ell$, hence the condition of EJR must be satisfied.

The core property drops the requirement of cohesiveness from EJR, and looks to prevent any group of agents from deviating to their mutual benefit. Does PAV satisfy this stronger requirement?

The Core

The following shows that PAV does not satisfy the core property.

Example: There are 15 candidates and 6 agents with a target committee size of $k = 12$.

- Agent 1 : $\{1, 2, 3, 4\}$
- Agent 2 : $\{1, 2, 3, 5\}$
- Agent 3 : $\{1, 2, 3, 6\}$
- Agent 4 : $\{7, 8, 9\}$
- Agent 5 : $\{10, 11, 12\}$
- Agent 6 : $\{13, 14, 15\}$

PAV returns the committee that omits candidates 4, 5 and 6.

Q: Why does this violate the core property?

Actually, none of the commonly studied rules satisfy the property. Moreover, it is known that no member of a broad class of rules can satisfy it.

Welfarist Rules

We now define the class of *welfarist rules*.

Definition (Welfarist Rule)

A welfare vector induced by a committee W captures the number of candidates in W that each agent approves:

$$welf_{\boldsymbol{A}}(W) = (|A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W|).$$

We say an approval-based multiiwinner rule $\mathcal{R}$ is *welfarist* if there exists a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ that maps welfare vectors to scores such that, for every election instance $(\boldsymbol{A}, k)$, we have:

$$\mathcal{R}(\boldsymbol{A}, k) = \text{argmax}_{W \in \mathcal{P}_k(C)} f(welf_{\boldsymbol{A}}(W)).$$

The Core versus Welfarism

Theorem (Peters and Skowron, 2020)

No welfarist rule satisfies the core property.

Note that the core captures the notion of proportionality based on 'power' instead of satisfaction, although it provides strong satisfaction guarantees (you will see a rule based on 'power' in the next lecture).

Open problem: Is the core always non-empty?

Can we get something more positive by moving to a restricted domain? Yes, let's touch back on party-list profiles.

Theorem (Brill et al., 2020)

In party-list election instances, PAV satisfies the core property.

Proof idea is similar to the proof showing PAV satisfies EJR.

References

1. Haris Aziz et al. (2017). "Justified Representation in Approval-Based Committee Voting". In: *Social Choice and Welfare* 48.2, pp. 461–485
2. Markus Brill et al. (2020). "Approval-based apportionment". In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 34. 02, pp. 1854–1861
3. Luis Sánchez-Fernández et al. (2017). "Proportional Justified Representation". In: *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI-2017)*, pp. 670–676
4. Dominik Peters and Piotr Skowron (2020). "Proportionality and the limits of welfarism". In: *Proceedings of the 21st ACM Conference on Economics and Computation*, pp. 793–794

Wrapping Up

Lecture Recap

Today's summary:

- ⋆ General multiwinner election model.
- ⋆ The core $\implies$ *EJR* $\implies$ *PJR* $\implies$ *JR*.
- ⋆ PAV satisfies EJR, but not the core.

Next time

Next time:

- ⋆ Simon's lecture on the computational aspects related to proportionality.
- ⋆ Reminder: To help prepare, you should watch videos on Computational Complexity (on the website), if you are unfamiliar with the topic.
- ⋆ Reminder: Read through the presentation topics (on the website) that you will discuss tomorrow.