Let's Agree to Agree: Targeting Consensus for Incomplete Preferences through Majority Dynamics

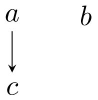
Simon Rey, together with Sirin Botan and Zoi Terzopoulou

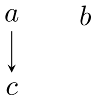
IJCAI 2022

1. Introduction

Deciding for an Online Platform

We have never used Bridge
AppEar is better than C-nnect





Deciding for an Online Platform

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Bridge is more stable than AppEar

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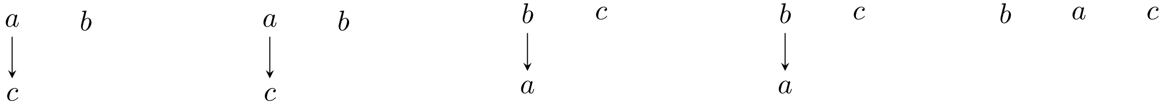
No opinion



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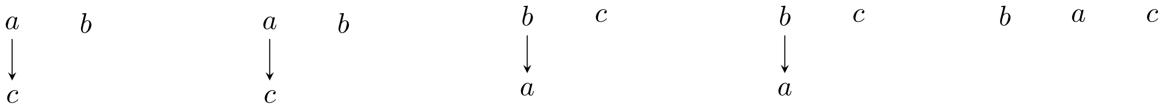

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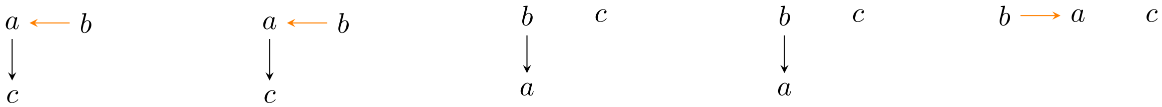

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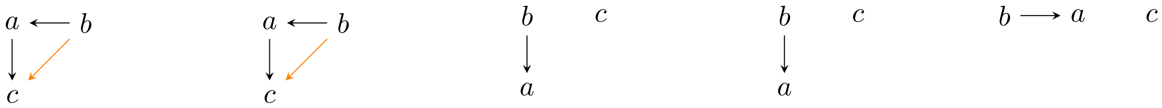

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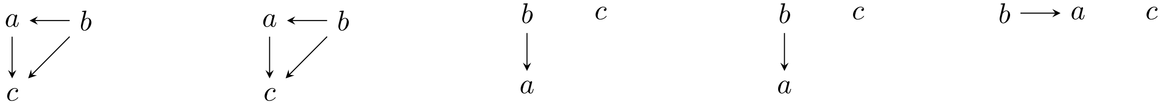

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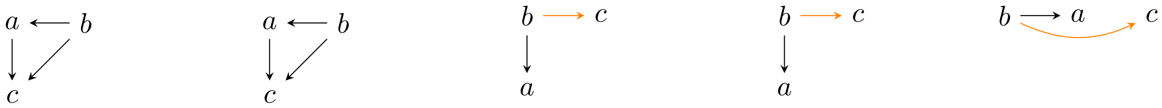

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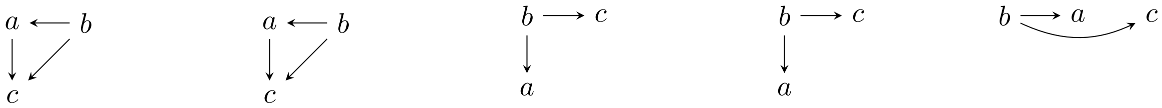

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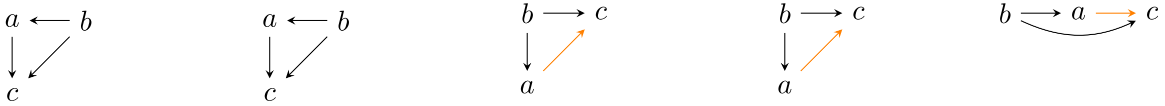

Deciding for an Online Platform

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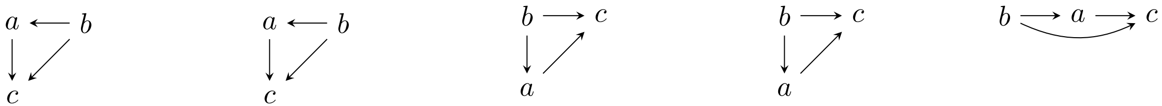

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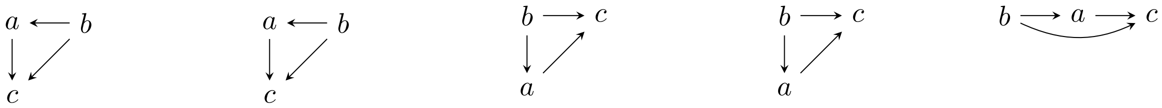

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Our goal is to study this dynamical process!

The Majority Dynamic

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Question of Interest

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What is consensus?

What kind of effects?

2. Condorcet Consensus

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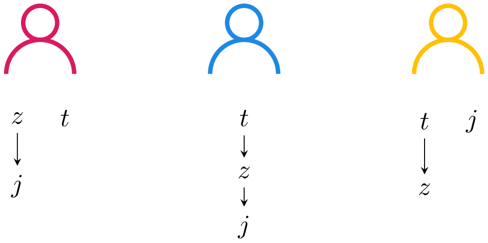

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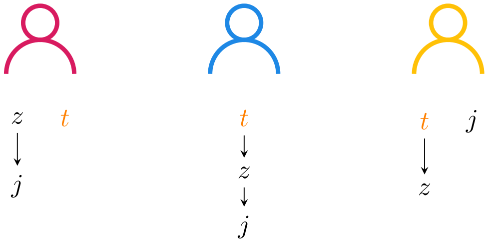

t against z: 2 for t 0 for z
t against j: 1 for t 0 for j

Condorcet Consensus

Preservation

Preserving Condorcet Consensus

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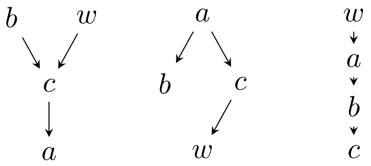

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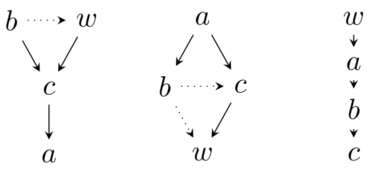

Updating on bc and bw

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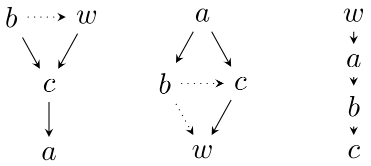

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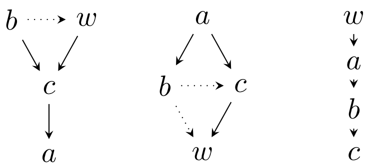
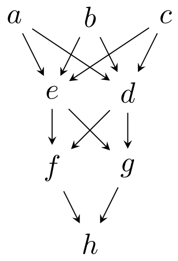


For 3 alternatives and less: Majority dynamic preserves existence of but not identity.

Strict Weak Orders

Strict Weak Orders: alternatives ranked in different levels, incomparabilities within levels



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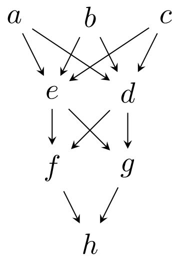

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- If we update on wa or aw then i will prefer w over a since w is a Condorcet winner ✓
- If we update on another pair leading to $a \succ_i^t w$. Then:

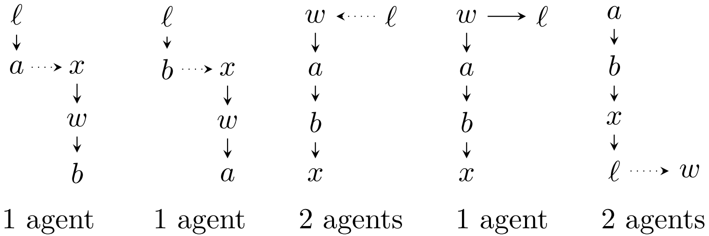
$$a \longrightarrow b \longrightarrow c \quad \cdots \quad z \longrightarrow w$$

This is impossible with strict weak preferences if i does not have opinion on a and w. ✓

Condorcet Consensus

Quality

A Condorcet loser can be turned into a Condorcet winner.



➥ Condorcet consensus is preserved (w initially and ℓ eventually) but the consensual alternative at the end used to be a Condorcet looser.

Condorcet Consensus

Controlling

From Preservation to Control

So far we focused on preserving consensus, i.e., universal guarantees that the majority dynamic does not harm consensus.

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What's next? Exploring what the decision maker can achieve by selecting a specific update order.

Positive and Negative Control

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Positively Controlling Condorcet Consensus

Positive Control: The majority dynamics *enables* positive Condorcet consensus control.

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↳ For a profile with a as initial Condorcet consensus, update according to $ab, ac, ad, ae \dots$

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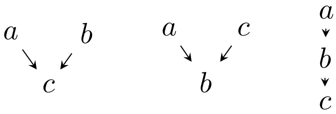
  - If there are no Condorcet winner at the end, we are done ✓
  - If $x \neq w$ is a Condorcet winner at the end, we have two update orders leading to two different Condorcet winners ✓

3. Other Consensus Notions

Unanimity-Based Consensus

Unanimity Undomiated



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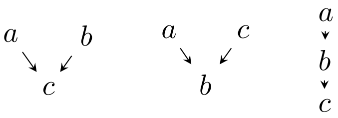
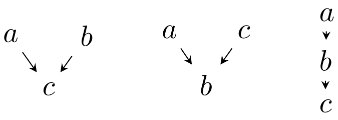
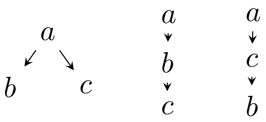


Consensus requires *unicity*

Unanimity Undomiated



Unanimity Dominant



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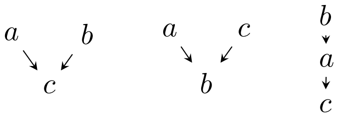
Majority Undomiated



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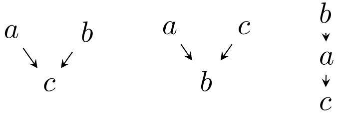

Majority should be *strict*

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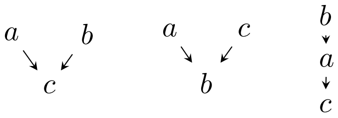

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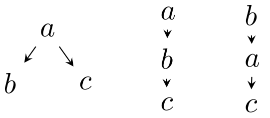

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Plurality-Based Consensus

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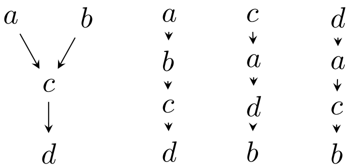

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Plurality-Based Consensus

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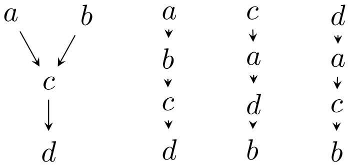

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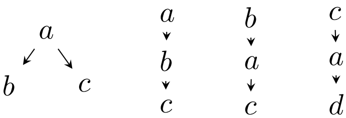
Plurality Dominant



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Majority Dominant	✓	✓	✗

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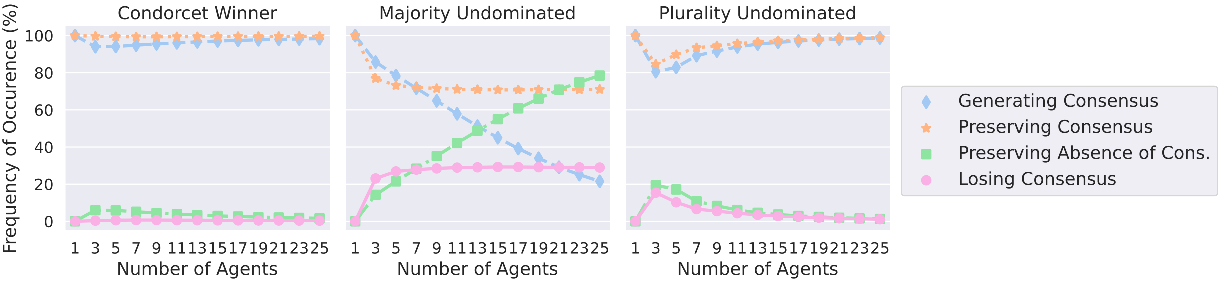
Majority Dominant	✓	✓	✗
Unanimity Undominated	✗ (✓)	✓	✓

Our Results

	Preserving consensus	Positive control	Negative control
Condorcet	✗ (✓)	✓	✓
Plurality Undominated	✗	✗	✗
Plurality Dominant	✗	✗	✗
Majority Undominated	✗	✓	✗
Majority Dominant	✓	✓	✗
Unanimity Undominated	✗ (✓)	✓	✓
Unanimity Dominant	✓	✓	✗

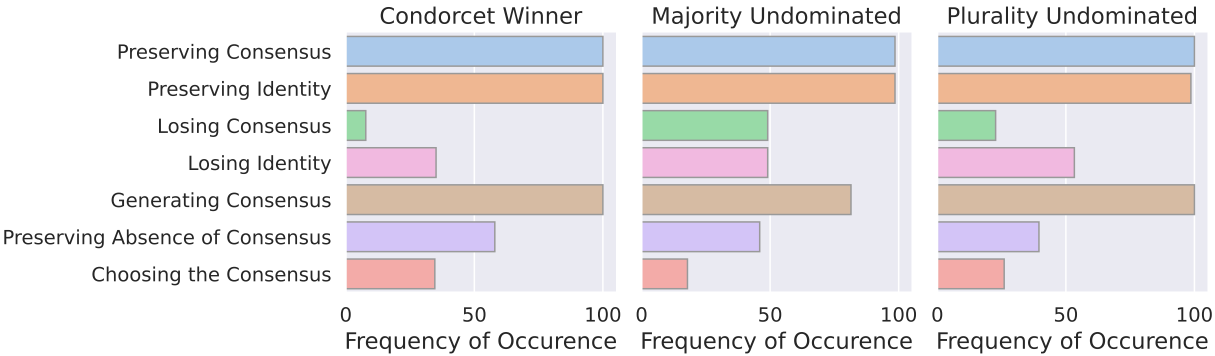
4. Experimental Analysis

Quantifying the Effects on Consensus



- Numbers of agents varying in 1, 3, 5, . . . , 25
- 5 000 000 random profiles each time (uniform distribution over $a \succ b$, $b \succ a$ and $a \sim b$, repeat until transitive)

Quantifying the Opportunities for Control



- 11 agents and 4 alternatives
- 20 000 random profiles
- All of the 46 080 possible update orders on each profile

5. Conclusion

Wrapping Up

What have we done? Studied the majority dynamic and the effects it can have on consensus for several consensus notions.

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What have we done? Studied the majority dynamic and the effects it can have on consensus for several consensus notions.

What is still to be done? Several ideas:

- Computational complexity of control problems (selecting the update order to achieve some goal)
- Computational complexity of good update orders (minimising number of updates, etc...)
- Guarantees about distance to consensus when it is not achieve
- And so many others...

Thanks

Sirin

Simon

Zoi

Do you still have questions?