Hardness Theory of Parameterized Complexity

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From Tractability to Intractability

During the first two lectures, we studied the class FPT that contains all the *fixed-parameter tractable* problems. These problems are tractable—solvable in polynomial time—when the value of the parameter is fixed. In that sense, FPT can be thought as an equivalent of P in classical complexity theory.

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Before going further into parameterized complexity, let's start with a remainder about intractability in classical complexity theory.

1. A Detour to Classical Complexity Theory

Definition: Complexity Class NP

NP is the class of all the languages $L \subseteq \Sigma^*$ for which there exists a Turing machine $\mathbb{M}$ (the verifier), a polynomial $p : \mathbb{N} \to \mathbb{N}$, and a constant $c \in \mathbb{N}$ such that:

- For all $x \in \Sigma^*$, we have:

$$x \in L \quad \text{if and only if} \quad \exists u \in \{0,1\}^{p(|x|)} \text{ (the certificate) such that } \mathbb{M}(x,u) = 1;$$

- $\mathbb{M}$ runs in time $\mathcal{O}(|x|^c)$ on every input $x \in \Sigma^*$.

While P was the class of all the problems decidable in polynomial time, NP is the class of all the problems for which we can *verify* a (potential) solution in polynomial time.

Non-Deterministic Turing Machines

<u>Definition</u>: Non-Deterministic Turing Machines

A *non-deterministic Turing machine* $\mathbb{M}$ is a of a Turing machine such that:

- It has two transition functions δ_1 and δ_2 (instead of only one);
- At each step, the one to be used is chosen non-deterministically;
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$\texttt{NP}$ can be equivalently defined through non-deterministic Turing machines.

Proposition: Another Characterization of $\texttt{NP}$

$\texttt{NP}$ is the class of all the languages $L \subseteq \Sigma^*$ for which there exists a *non-deterministic* Turing machine $\mathbb{M}$ running in *polynomial* time and *deciding* L.

The Class coNP

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coNP can also be seen as the class of all the problems for which checking whether something is *not* a solution is an NP problem.

Proposition: Complexity Class coNP

A language $L \subseteq \Sigma^*$ is in coNP if and only if $\overline{L} = \{x \in \Sigma^* \mid x \notin L\}$ is in NP.

Some Examples

Is Pareto-Optimal	
Instance:	A set of items $\mathcal{I}$, a set of n agents $\mathcal{N}$, n utility functions $u_i : \mathcal{I} \to \mathbb{N}$, and an allocation $\pi : \mathcal{N} \to 2^{\mathcal{I}}$ such that all items are allocated and no item is allocated to several agents (a partition of the items)
Question:	Is the allocation π Pareto-optimal, i.e., there is no other allocation π' such that all agents are better off in π' and at least one agent is strictly better off?

Max-Approval Participatory Budgeting	
Instance:	A set of projects $\mathcal{P}$, a cost function $c : \mathcal{P} \to \mathbb{N}$, a budget limit $B \in \mathbb{N}$, a set of agents $\mathcal{N} = \{1, \ldots, n\}$, n approval ballots $A_i \subseteq \mathcal{P}$ and a parameter $k \in \mathbb{N}$
Question:	Is there a budget allocation $\pi \subseteq \mathcal{P}$ with $\sum_{p \in \pi} c(p) \leq B$ and such that $\sum_{i \in \mathcal{N}} \lvert \pi \cap A_i \rvert \geq k$?

↳ Who is where?

NP-completeness

The hardness theory in classical complexity theory is based on the idea that several problems are *equivalent* in terms of how hard they are to solve. The formalization of this idea is based on *polynomial time reductions*.

A language $L_1 \subseteq \Sigma^*$ is *polynomial time reducible* to another language $L_2 \subseteq \Sigma^*$ if there exists a polynomial time computable function $f : \Sigma^* \to \Sigma^*$ (the reduction) such that for all $x \in \Sigma^*$, we have: $x \in L_1 \iff f(x) \in L_2$.

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A language $L \subseteq \Sigma^*$ is NP-*hard* if every L' in NP is polynomial time reducible to L.
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To show that a language L is NP-hard we start from a language L_0 in NP (Max-Approval Participatory Budgeting for instance, or, historically, SAT) and provide a polynomial time reduction showing how to *embed* L_0 in L so that if one can decide L, one would also decide L_0. In that sense, all NP-complete languages are *equally hard* to solve: solving one implies solving all of them (since reductions are transitive).

2. Fixed-Parameter Tractable Reduction

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Definition: FPT-Reduction

An *FPT-reduction* from a parameterized language $L_1 \subseteq \Sigma^* \times \mathbb{N}$ and to another $L_2 \subseteq \Gamma^* \times \mathbb{N}$ is a mapping $f : \Sigma^* \times \mathbb{N} \to \Gamma^* \times \mathbb{N}$ such that:

1. $\langle x, k \rangle \in L_1$ if and only if $f(x, k) \in L_2$;
2. $k' \leq g(k)$ for some computable function g for k' such that $\langle x', k' \rangle = f(x, k)$;
3. f is computable by an FPT algorithm (with respect to k).

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Condition 2 ensures that the class FPT is closed under FPT-reduction.

Proposition: Two Facts About FPT-Reduction

The relation between languages derived from FPT-reductions is transitive.

If there is an FPT-reduction from a parameterized language $L_1 \subseteq \Sigma^* \times \mathbb{N}$ to another parameterized language $L_2 \in \Gamma^* \times \mathbb{N}$ that is in FPT, then L_1 is in FPT.

A Parameterized Equivalent of NP?

FPT can be interpreted as the parameterized counterpart of P (recall that for a language L in P, all of its parameterizations are in FPT). Now that we have a suitable notion of parameterized reduction, can we can try to define an equivalent of NP in parameterized complexity theory.

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↪ We will see there is not a single class that resemble NP in parameterized complexity theory but rather a whole *hierarchy* of them. Let's begin!

3. Parameterized NP?

Parameterized NP?

First Attempt: paraNP

Non-Deterministic Parameterized Classes

To move from $\texttt{P}$ to $\texttt{NP}$, we plugged in non-deterministic Turing machines. Let's do the same with $\texttt{FPT}$, we obtain the class $\texttt{paraNP}$.

DEFINITION: PARAMETERIZED COMPLEXITY CLASS $\texttt{paraNP}$

$\texttt{paraNP}$ is the class of all the parameterized languages $L \subseteq \Sigma^* \times \mathbb{N}$ for which there exists a *non-deterministic* Turing machine $\mathbb{M}$, a constant $c \in \mathbb{N}$ and a computable function f such that, for all pairs $\langle x, k \rangle \in L$:

- $\mathbb{M}$ runs in time $f(k)|x|^c$ on $\langle x, k \rangle$;
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It is clear that for all languages L in NP, all the parameterized languages $L' \subseteq L \times \mathbb{N}$ are in paraNP. In that sense paraNP is an equivalent to NP. Similarities are even stronger:

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Proposition:

$\texttt{FPT} = \texttt{paraNP}$ if and only $\texttt{P} = \texttt{NP}$.

paraNP-Complete Parameterized Languages

Before giving the main result about paraNP, we need two more things:

- The fact that paraNP is closed under FPT-reductions;
- The k-slice of a parameterized language $L \subseteq \Sigma^* \times \mathbb{N}$ defined as:

$$L_k = \{x \in \Sigma^* \mid \langle x, k \rangle \in L\}.$$

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Now the main result:

THEOREM:

Let $L \subseteq \Sigma^* \times \mathbb{N}$ be a non-trivial parameterized language in paraNP. The following statements are equivalent:

- L is paraNP-complete under FPT-reduction;
- There exist $\ell \in \mathbb{N}_{>0}$ and $k_1, \ldots, k_\ell \in \mathbb{N}$ such that $L_{k_1} \cup \ldots \cup L_{k_\ell}$ is NP-complete (under polynomial time reductions).

Implications and Interpretations

COROLLARY:

A non-trivial parameterized language in paraNP with at least one NP-complete slice is paraNP-complete under FPT-reduction.

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Let's try to find a better counterpart for NP in the parameterized world!

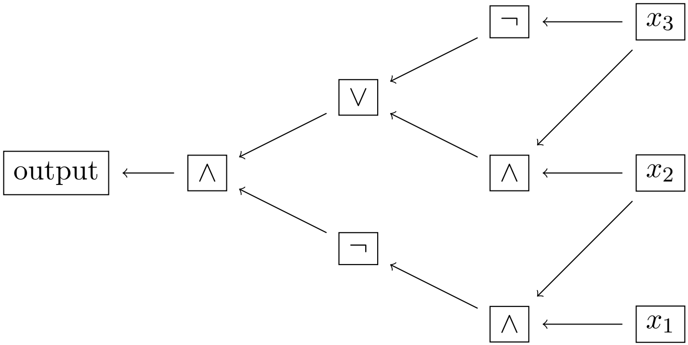
Parameterized NP?

Another try: the W Hierarchy

Boolean Circuit

Definition: Boolean Circuit

A *boolean circuit* is a directed acyclic graph whose nodes are labeled either with a Boolean constant ($\top$ or $\bot$), a propositional variable or a Boolean operator ($\wedge$, $\vee$, $\neg$). There is also a specified output node with outdegree 0.



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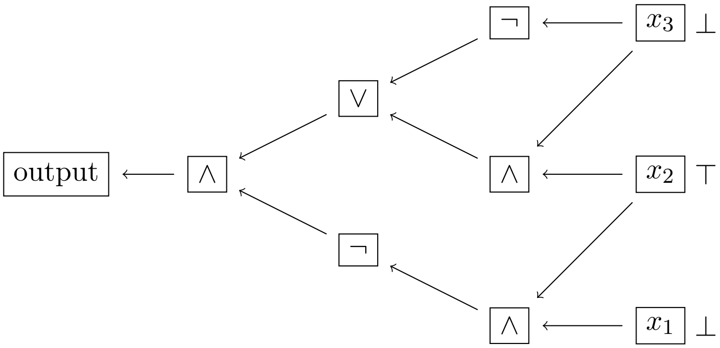

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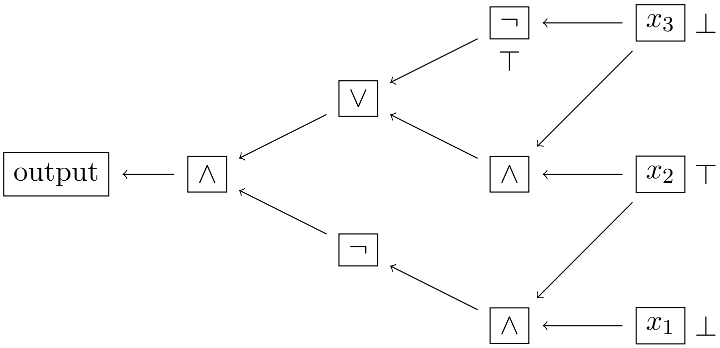

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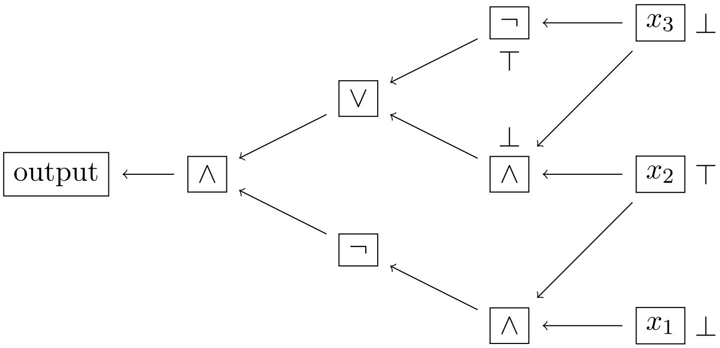

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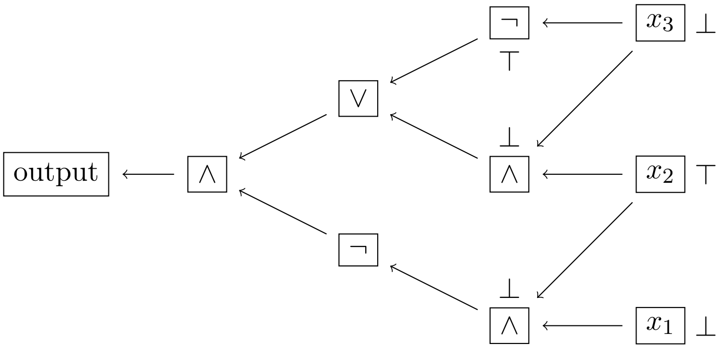

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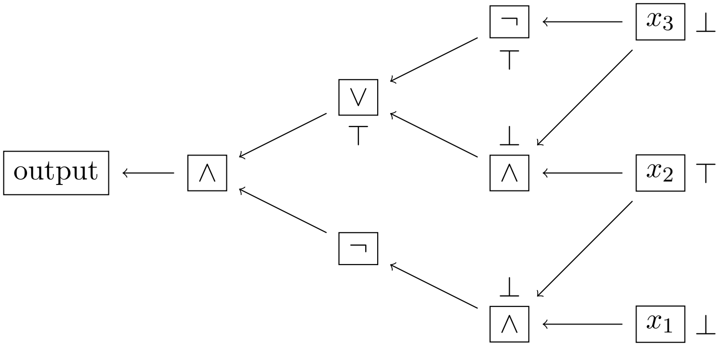

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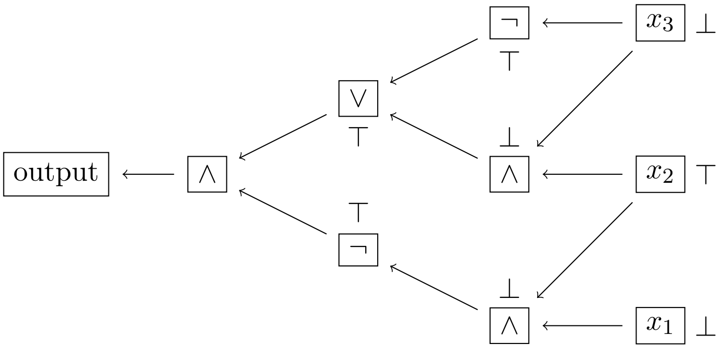

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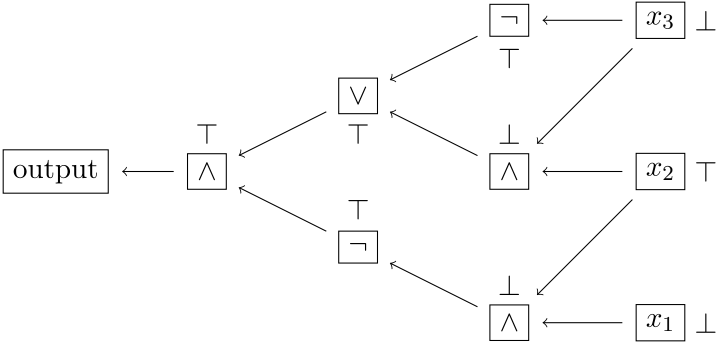

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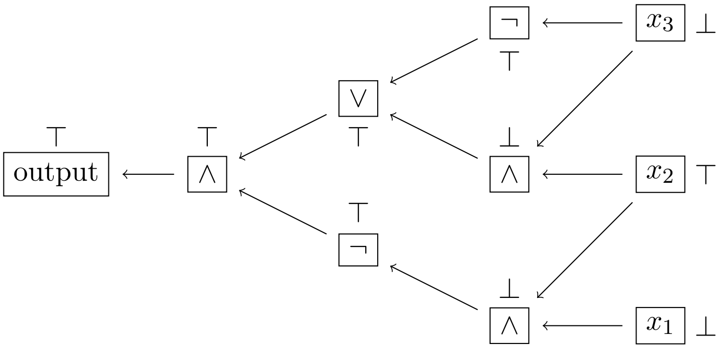
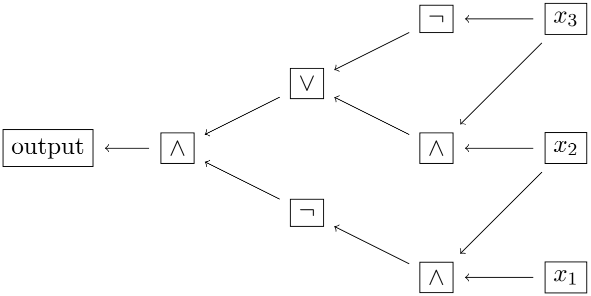


Properties of Boolean Circuits

We will need the following definitions about boolean circuits.

- A *formula* is a Boolean circuit in which all gates have outdegree at most 1.
- The *depth* of a Boolean circuit is the length of a longest path from a variable node to the output node.
- The *weft* of a Boolean circuit is the largest number of large gates (with indegree at least 3) on any path from a variable node to the output node.



↪ Is the previous Boolean circuit a formula? What is its depth? And its weft?

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Finally, an assignment maps each variable to a truth value ($\bot$ or $\top$). A assignment *satisfies* the Boolean circuit C if after propagating the truth values, the output node is set to $\top$. An assignment has *weight k* if it sets exactly k variables to $\top$.

A Family of Classes: $\mathtt{W}[t]$

The family of classes $\mathtt{W}[t]$ is defined with respect to the following parameterized language.

WSat($\mathcal{C}$)	
Instance:	A Boolean circuit $C \in \mathcal{C}$ and $k \in \mathbb{N}$
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The parameterized complexity class $\texttt{W}[t]$, for $t \in \mathbb{N}_{>0} \cup \{SAT, \texttt{P}\}$, is defined as:

$$\begin{aligned} \texttt{W}[t] &= [\{\text{WSAT}(CIRC_{t,u}) \mid u \geq 1\}]^{\texttt{FPT}}, \\ \texttt{W}[SAT] &= [\text{WSAT}(FORM)]^{\texttt{FPT}}, \\ \texttt{W}[\texttt{P}] &= [\text{WSAT}(CIRC)]^{\texttt{FPT}}, \end{aligned}$$

where $[\mathcal{S}]^{\texttt{FPT}}$ is the transitive closure of $\mathcal{S} \subseteq 2^{\Sigma^* \times \mathbb{N}}$ under $\texttt{FPT}$-reductions.

Restricted Non-Determinism and $\mathtt{W[P]}$

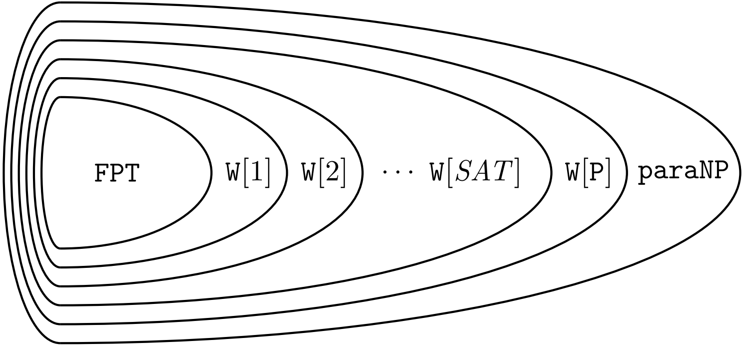
$\mathtt{W[P]}$ can be alternatively characterized with non-deterministic Turing machines that have only limited access to non-determinism.

A non-deterministic Turing machine $\mathbb{M}$ is *k-restricted* if there are two functions $f, g : \mathbb{N} \to \mathbb{N}$ and a constant $c \in \mathbb{N}$ such that on every input $\langle x, k \rangle \in \Sigma^* \times \mathbb{N}$, the machine $\mathbb{M}$ runs in at most $f(k)|x|^c$ steps of which at most $g(k) \log(|x|)$ are non-deterministic.

Proposition:

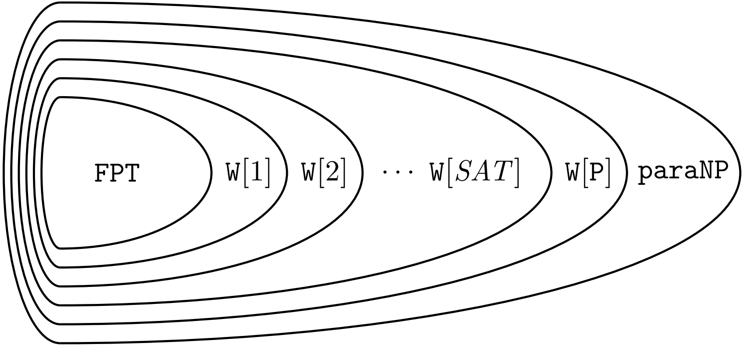
$\mathtt{W[P]}$ is the class of all the parameterized languages $L \subseteq \Sigma^* \times \mathbb{N}$ that can be decided by a non-deterministic *k-restricted* Turing machine $\mathbb{M}$.

Current Taxonomy of Parameterized Complexity Classes



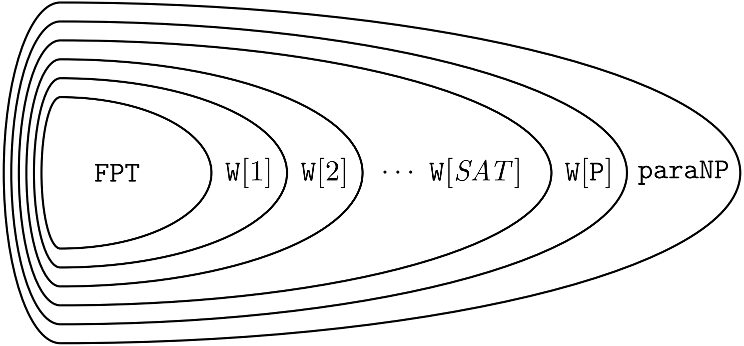
<u>Proposition</u>:

$\mathtt{FPT} \subseteq \mathtt{W}[1] \subseteq \mathtt{W}[2] \subseteq \cdots \subseteq \mathtt{W}[SAT] \subseteq \mathtt{W}[\mathrm{P}] \subseteq \mathtt{paraNP}$.



All the inclusions in the figure above are *believed* to be strict. However, it seems really hard to show this. In particular, for any $t \in \mathbb{N} \cup \{SAT, \mathtt{P}\}$ if $\mathtt{FPT} \neq \mathtt{W}[t]$, then also $\mathtt{P} \neq \mathtt{NP}$ (since $\mathtt{P} = \mathtt{NP}$ if and only $\mathtt{FPT} = \mathtt{paraNP}$). Whether the converse also holds is an open problem.

Towards Intractability



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↪ Let's now look at a class of parameterized languages that we can *prove* is *strictly larger* than FPT (diagonalization is a great tool!).

4. Provable Fixed-Parameter Intractability

Non-Uniform XP

Let's come back to the idea of having polynomial slices that we mentioned in paraNP.

DEFINITION: PARAMETERIZED COMPLEXITY CLASS $\mathtt{XP}_{nu}$

$\mathtt{XP}_{nu}$ is the class of all the parameterized languages $L \subseteq \Sigma^* \times \mathbb{N}$ whose slices L_k for $k \in \mathbb{N}_{>0}$ are *all in* P.

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$\mathtt{XP}_{nu}$ is the class of all the parameterized languages $L \subseteq \Sigma^* \times \mathbb{N}$ whose slices L_k for $k \in \mathbb{N}_{>0}$ are *all in* P.

This class is *non-uniform* in the sense that it contains undecidable problems.

PROOF: Consider an undecidable language $Q \subseteq 1^*$ and its parameterized version $Q^{para} = \{\langle x, k\rangle \mid x \in Q, k = \max\{1, |x|\}\}$. The k-slide of Q^{para} is then $Q_k^{para} = \{x \in Q \mid |x| = k\}$. That is, $Q_k^{para} = \emptyset$ if $1^k \notin Q$ and $Q_k^{para} = \{1^k\}$ otherwise. This is trivially decidable in polynomial time. Q^{para} is thus in $\mathtt{XP}_{nu}$.

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↳ This is not so nice, let's get rid of the non-uniformity!

(Uniform) XP

Definition: Parameterized Complexity Class XP

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- $\mathbb{M}$ runs in time $|x|^{f(k)} + f(k)$ on $\langle x, k \rangle$;
- $\langle x, k \rangle \in L$ if and only if $\mathbb{M}(\langle x, k \rangle) = 1$.

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While we have seen parameterized classes that resembles NP, XP plays the role of EXP in parameterized complexity theory. In the following, we will show that XP *is a strict superset of* FPT, mimicking the fact that $\mathtt{P} \subsetneq \mathtt{EXP}$.

An XP-Complete Language

We first show that the following parameterized language is XP-complete under FPT-reductions.

Exp-DTM-Halt	
Instance:	A Turing machine $\mathbb{M}$, $n \in \mathbb{N}$ in unary and $k \in \mathbb{N}$
Parameter:	k
Question:	Does $\mathbb{M}$ accept the empty string ϵ in at most n^k steps?

To show XP-completeness, we will first show that the language *Exp-DTM-Halt is in* XP by presenting an algorithm solving it in time $|x|^{f(k)} + f(k)$. In a second time, we will show that the language is XP-*hard* by presenting an FPT-reduction from an arbitrary problem in XP to Exp-DTM-Halt.

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Proof: An algorithm witnessing membership in XP simply *simulates* $\mathbb{M}$ on ϵ for n^k steps and output the result of the simulation. This is possible since an *efficient universal Turing machine* exists.

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Proof: Let's show XP-*hardness* now. Take any parameterized language $L \subseteq \Sigma^* \times \mathbb{N}$ in XP. Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function and $\mathbb{M}$ a Turing machine *deciding* if $\langle x, k\rangle \in L$ in time $|x|^{f(k)} + f(k)$.

Consider another Turing machine $\mathbb{M}'$ which first *writes* $\langle x, k\rangle$ on its input tape and then *simulates* $\mathbb{M}$ *on* $\langle x, k\rangle$.

Assume without loss of generality that for some computable $g : \mathbb{N} \to \mathbb{N}$, $\mathbb{M}'$ needs at most $(|x| + 2)^{g(k)}$ steps on input $\langle x, k\rangle$.

The function that maps any $\langle x, k\rangle \in L$ to an instance $\langle \mathbb{M}'(\langle x, k\rangle), |x| + 2, g(k)\rangle$ of Exp-DTM-Halt is an FPT-*reduction from* L *to* Exp-DTM-Halt.

The previous result is particularly interesting for us as it allows to show that FPT is a strict subset of XP. This entails that XP-hard languages *cannot* be solved in FPT time.

Theorem:

$\mathtt{FPT} \subsetneq \mathtt{XP}$.

Provable Fixed-Parameter Intractability

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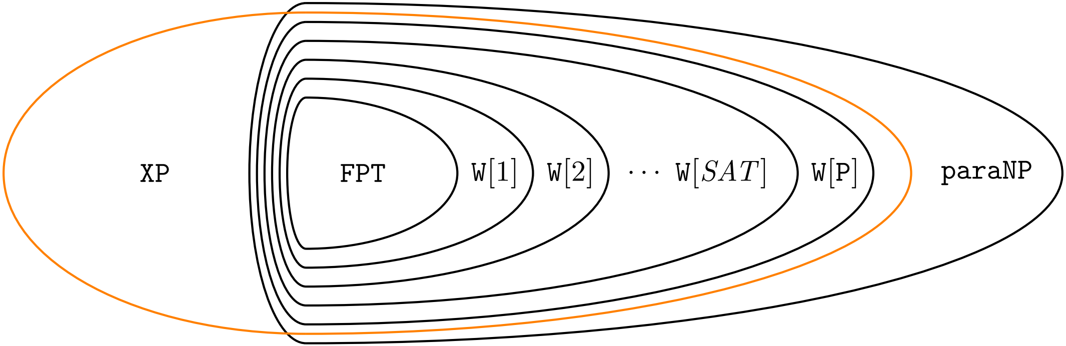
<u>THEOREM</u>:

$\texttt{FPT} \subsetneq \texttt{XP}$.

<u>PROOF</u>: Obviously $\texttt{FPT} \subseteq \texttt{XP}$. Assume now that EXP-DTM-HALT is in FPT. Then, for some $c \in \mathbb{N}$ all the slices of EXP-DTM-HALT are solvable in $\texttt{DTIME}[n^c]$. In particular, the $(c+1)$-slice also is. It implies that $\texttt{DTIME}[n^{c+1}] \subseteq \texttt{DTIME}[n^c]$ which contradicts the *time hierarchy theorem* (see below).

The time hierarchy theorem states that for two time-constructible functions $f, g : \mathbb{N} \to \mathbb{N}$, if $f(n)\log(f(n))$ is $o(g(n))$, then $\texttt{DTIME}[f(n)] \subsetneq \texttt{DTIME}[g(n)]$. Roughly speaking, this says that we can solve strictly more problems when allowing extra running time.

Final Taxonomy of What we Have Seen

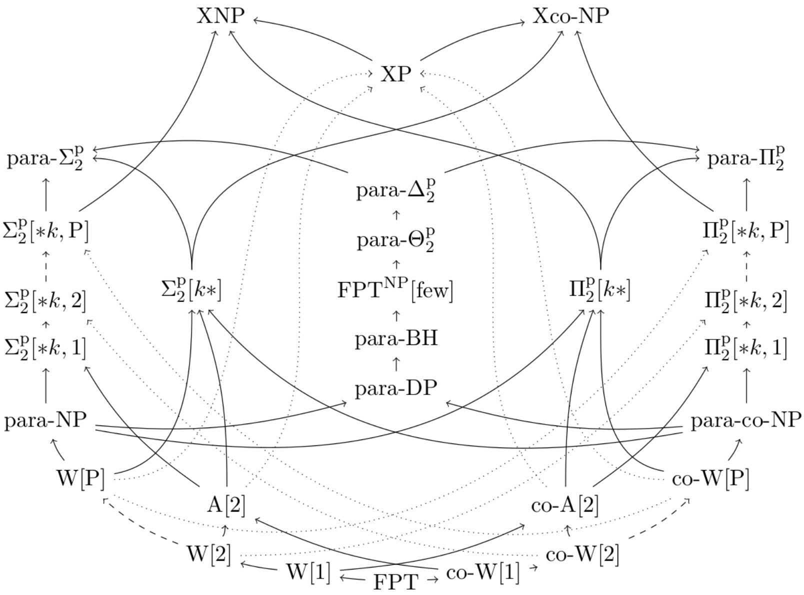


Proposition:

$\mathtt{FPT} \subseteq \mathtt{W}[1] \subseteq \mathtt{W}[2] \subseteq \cdots \subseteq \mathtt{W}[SAT] \subseteq \mathtt{W}[\mathtt{P}] \subseteq \mathtt{paraNP} \cap \mathtt{XP}$.

5. Conclusion

Want More?


XNP
Xco-NP
XP
para-Σ_2^p
para-Π_2^p
para-Δ_2^p
$\Sigma_2^p[*k, P]$
$\Pi_2^p[*k, P]$
para-Θ_2^p
$\Sigma_2^p[*k, 2]$
$\Sigma_2^p[k*]$
FPT^{NP}[few]
$\Pi_2^p[k*]$
$\Pi_2^p[*k, 2]$
$\Sigma_2^p[*k, 1]$
para-BH
$\Pi_2^p[*k, 1]$
para-DP
para-NP
para-co-NP
W[P]
co-W[P]
A[2]
co-A[2]
W[2]
co-W[2]
W[1]
FPT
co-W[1]

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In the last lecture of the week, Ronald will tell you about some more advanced techniques about lower bounds for kernelization.