Parameterized Complexity Theory and its Applications to Social Choice

Kernelization Lower Bounds

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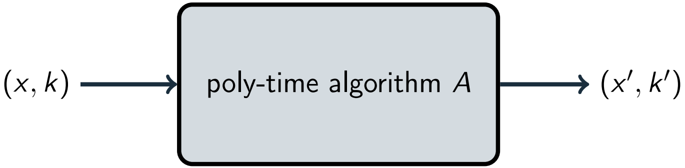
Plan for today

- Polynomial kernels
- Possible Winner for k-Approval
- Machinery to show lower bounds on kernels

Recap: kernelization



$|x'| \leq g(k)$

$k' \leq g(k)$

$(x, k) \in Q$ if and only if $(x', k') \in Q$

- A *kernelization (algorithm)* (or *kernel*) is a poly-time reduction from Q to itself, such that the output is of size $\leq g(k)$

Recap: kernelization (ct'd)

Theorem ("FPT iff kernelization")

A parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$ is fixed-parameter tractable if and only if it admits a kernelization algorithm.

Example: kernelization for Vertex Cover

- Kernelization algorithm for Vertex Cover:

  (a) Repeat the following two rules until you reach a fixpoint.

    1. If $k > 0$, and if there is a vertex v with degree $> k$, remove v and decrease k by one.

    2. If there is an isolated vertex v, remove v.

  (b) If the resulting graph G has more than k^2 edges, return a trivial no-instance.

  (c) Otherwise, return (G, k)

- Why are rules 1 and 2 and step (b) correct?

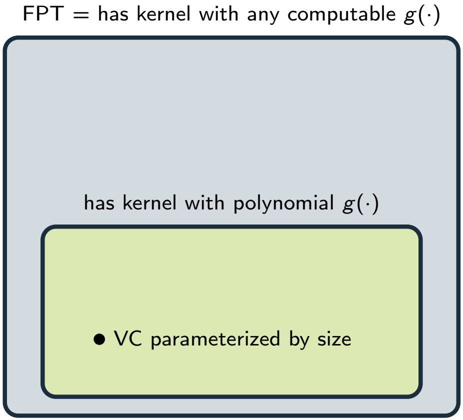
- Why does this satisfy the requirements of a kernelization?

Structure inside FPT: size of kernels

- By distinguishing different classes of functions $g(\cdot)$, we can group problems in FPT into different subclasses

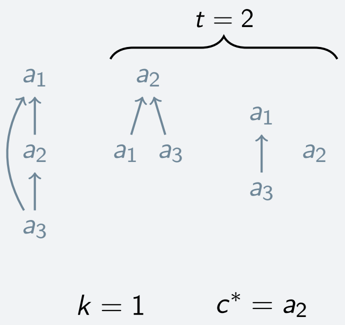
Definition (polynomial kernel)

A kernelization is called a *polynomial kernelization* if the function $g(\cdot)$ bounding the size of the output is a polynomial function.



Example: Possible Winner for k-Approval

- Possible Winner for k-Approval
  - *Input:* A set C of candidates, $k \in \mathbb{N}$, a set of orderings over C (the votes), of which t are partial orders and the rest are linear orders, and some $c^* \in C$.
  - *Parameter:* k, t
  - *Question:* Can we complete the partial orders so that c^* becomes a winner under the k-Approval rule?



Betzler, N. *On problem kernels for possible winner determination under the k-approval protocol.* Proceedings of the International Symposium on Mathematical Foundations of Computer Science (MFCS). 2010.

Kernelization for Possible Winner for k-Approval

- There is a kernelization algorithm for Possible Winner for k-Approval, parameterized by k and t
  - **Rule 1:** fix c^* as high as possible in each partial ranking
  - *(.. more rules and more details ..)*
- The bound $g(k)$ on the size of the kernelization is some ‘highly exponential’ function

? Can we improve this to a polynomial kernelization?

Can we prove whether some problem does not admit poly-size kernels?

- Yes, we can! ☺

Bodlaender, H.L., Downey, R.G., Fellows, M.R., Hermelin, D.: *On problems without polynomial kernels.* J. Computer and System Sciences 75(8), 423–434 (2009)

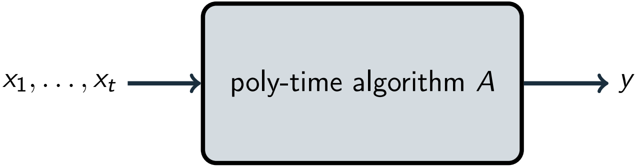
OR-Compositionality

$$(x_1, k), \ldots, (x_t, k) \longrightarrow \boxed{\text{poly-time algorithm } A} \longrightarrow (y, k') \qquad k' \leq \text{poly}(k)$$

$$(x_i, k) \in Q \text{ for some } i \in \{1, \ldots, t\} \quad \text{if and only if} \quad (y, k') \in Q$$

- An *OR-composition algorithm* for Q is a poly-time reduction from an "OR-variant of Q" to Q, such that $k' \leq \text{poly}(k)$

OR-Distillation



$|y| \leq \text{poly}(\max |x_i|)$

$x_i \in L$ for some $i \in \{1, \ldots, t\}$ if and only if $y \in R$

- An *OR-distillation of L into R* is a poly-time reduction from an "OR-variant of L" to R

In NP $\wedge$ composition $\wedge$ poly-size kernel $\rightarrow$ distillation

Theorem

Let Q be a parameterized problem that has an OR-composition algorithm, whose unparameterized version is in NP*. If Q has a polynomial kernel, then there is a distillation from the unparameterized version of Q to* SAT*.*

- Distillation algorithm:
  1. Take inputs $x_1, \ldots, x_t$ for the unparameterized version of Q, and consider the corresponding parameters $k_1, \ldots, k_t$
  2. For each $1 \leq \ell \leq \max k_i$, consider all inputs x_i for which $k_i = \ell$, and apply the composition on this subsequence of (x_i, k_i)'s—giving $(y_1, k_1'), \ldots, (y_r, k_r')$
  3. Then apply the polynomial kernel on each of $(y_1, k_1'), \ldots, (y_r, k_r')$, giving $(z_1, k_1''), \ldots, (z_r, k_r'')$
  4. Compute propositional formulas φ_i that each are satisfiable if and only if $(z_i, k_i'') \in Q$, and return $\bigvee_i \varphi_i$

Bring in the big guns..

Theorem (Fortnow, Santhanam, 2011)

If an NP*-hard problem L admits an OR-distillation into some problem R, then* $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

Theorem (Karp, Lipton, 1980)

If $\mathsf{NP} \subseteq \mathsf{coNP/poly}$*, then the Polynomial Hierarchy (PH) collapses.*

- So assuming that the PH does not collapse, any NP-complete problem that admits an OR-composition does not admit a polynomial kernel.

Example of an OR-composition: Longest Path

- Longest Path, parameterized by k:
  - *Input:* (G, k), where G is an undirected graph, and $k \in \mathbb{N}$
  - *Parameter:* k
  - *Question:* Is there a simple path in G of length at least k?

- Longest Path admits an OR-composition
  -

1. Take inputs $(G_1, k), \ldots, (G_t, k)$
2. Output $(G_1 \,\dot\cup\, \cdots \,\dot\cup\, G_t, k)$

Composition for Possible Winner for k-Approval

Theorem (Betzler, 2010)

Possible Winner for k-Approval, parameterized by k and the number t of partial votes, admits an OR-composition.

Corollary (Betzler, 2010)

If Possible Winner for k-Approval, parameterized by k and the number t of partial votes, admits a polynomial kernel, then the PH collapses.

More general framework

- Weaker assumptions:
  - We required the inputs $(x_1, k), \ldots, (x_t, k)$ to have the same parameter value

    we can slightly relax this to: the inputs must be equivalent according to some equivalence relation $\mathcal{R}$ (with some nice properties)
  - Composition algorithms can be into another problem R
  - It also works for AND- instead of OR-compositions (and -distillations).
- Stronger conclusions:
  - Rule out compression algorithms: polynomial kernelization into another problem

Recap

- Polynomial kernels
- Possible Winner for k-Approval
- Machinery to show lower bounds on kernels
  - OR-compositions, OR-distillations
  - in NP + OR-composition + poly-size kernel $\Rightarrow$ OR-distillation
  - OR-distillation + NP-hard $\Rightarrow$ PH collapses