Fairness in Long-Term Participatory Budgeting

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Abstract

Participatory Budgeting (PB) processes are usually designed to span several years, with referenda for new budget allocations taking place regularly. This paper presents a first formal framework for long-term PB, based on a sequence of budgeting problems as main input. We introduce a theory of fairness for this setting, focusing on three main concepts that apply to types (groups) of voters: (i) achieving equal welfare for all types, (ii) minimizing inequality of welfare (as measured by the Gini coefficient), and (iii) achieving equal welfare in the long run. For different notions of welfare, we investigate under which conditions these criteria can be satisfied, and analyze the computational complexity of verifying whether they hold.

1 Introduction

Participatory Budgeting (PB) is a democratic tool in which citizens are asked their opinion on how to spend a public budget [Cabannes, 2004; Shah, 2007]. This process is now applied , which was invented in Brazil, is now used in many cities all around the world [Dias *et al.*, 2019]. The way it is precisely organized differs from place to place but generally the same two-stage structure is adopted [Shah, 2007]: first citizens propose projects and then they vote on these projects. Based on these votes, a set of projects is chosen that can be implemented with the available budget. Importantly, PB is usually planned to run for several years. For instance, a participatory budgeting process in Paris spanned 6 years (from 2014 to 2020) [City of Paris, 2020], and New York runs an ongoing program since 2011 [New York City Council, 2020]. The general idea of PB is to establish it as a regular, ongoing process for sustained citizen participation.

Even though PB has received substantial attention in recent years through the lens of (computational) social choice [Aziz and Shah, 2020], its formalizations generally consider PB as a one-shot process. This assumption significantly limits the scope of an analysis. In particular, it disregards the possibility of achieving fair outcomes *over time*, although a fair solution may be impossible to obtain in individual PB instances. The main purpose of our work is to close this gap: we introduce *perpetual participatory budgeting*, a formal framework that encompasses key characteristics of long-term PB.

The long-term perspective of perpetual PB leads to conceptual challenges but brings notable advantages. To highlight the potential of this approach, we introduce and study notions of fairness in this setting and analyze to which extent strong fairness guarantees can be achieved in long-term processes. We are mainly concerned with fairness towards *types* of voters. A type is a pre-defined subset of voters, for example all voters in a certain district or socio-demographic groups (e.g., age, education, income). Furthermore, to be able to speak about fairness, we have to specify how we measure the welfare of types. We consider three main forms of welfare: The first is *satisfaction*, which intuitively corresponds to the agreement between a voter's ballot and the chosen projects, weighted by cost. The satisfaction of a type is the average satisfaction of its voters. The second welfare notion is *relative satisfaction*, which is similar to satisfaction but measures the satisfaction relative to the voter's maximally achievable satisfaction. The third is the *share* of a type, which is the money spent on satisfying this type. It is natural to require that a type's share is proportional to its size.

In a first step, we define a very strong fairness criterion by requiring that all types achieve the same welfare. This is not only unachievable for obvious reasons in single-round PB, but we can show that there are arbitrary long perpetual PB instances where equal welfare remains unachievable. However, while equal welfare is often unattainable, different sets of projects can lead to vastly different distributions of welfare. Thus, as a second fairness criterion, we use the Gini index, a well-known inequality measure for income, to measure inequality with respect to welfare. This measure can be used, e.g., to analyze real-world budget allocations. In this paper, we take a computational approach: given a perpetual PB instance, we can use the Gini index as an optimization goal and search for the least unequal budget allocation. We show that testing for the optimality of a solution is already co-NP-complete, even in simple settings.

As a third fairness criterion we require that all types have the same welfare in the long run, i.e., they are asymptotically equal. Our main result for this fairness criterion is that it is always possible to achieve equal relative satisfaction in the limit if there are only two types. In contrast, equal shares and

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equal-satisfaction are impossible to achieve even in the long run, in particular due to inhomogeneous types. It remains an interesting open problem whether equal relative satisfaction can be guaranteed in the limit for any number of types.

To sum up, our paper contains two main contributions: (i) the framework of perpetual participatory budgeting and (ii) the analytic and computational study of three fairness criteria in this framework. These (strong) fairness criteria cannot be guaranteed in a single round of PB and thus necessitate our perpetual setting.

Related Work. The standard PB setting from the perspective of computational social choice has been extensively described [Lu and Boutilier, 2011; Talmon and Faliszewski, 2019; Aziz and Shah, 2020], and has then been extended in several directions [Jain *et al.*, 2020; Rey *et al.*, 2020; Shapiro and Talmon, 2017; Benadè *et al.*, 2020; Baumeister *et al.*, 2020; Peters *et al.*, 2020; Skowron *et al.*, 2020; Laruelle, 2021]. The main focus of prior works were normative properties: e.g., monotonicity [Talmon and Faliszewski, 2019], strategy-proofness [Goel *et al.*, 2019; Freeman *et al.*, 2019], the core property [Fain *et al.*, 2016], and proportionality [Aziz *et al.*, 2018]. A recent study of district fairness [Hershkowitz *et al.*, 2021] investigates whether a city-wide PB can guarantee each district the social welfare they would have had by running a district-wide PB. While we consider districts (called *types*), our notions of fairness differ.

None of the aforementioned works consider participatory budgeting as a repeating, ongoing process. We are only aware of one exception in which the outcome of the previous year is taken into account to compute a budget allocation [Shapiro and Talmon, 2017] (only for tie-breaking however). We take previous rounds of PB into account in a more comprehensive fashion. Finally, let us mention that our "perpetual" perspective has been also considered in classical voting [Lackner, 2020] and utility aggregation [Freeman *et al.*, 2017; Freeman *et al.*, 2018].

2 Motivating Example

Let us begin with an example that demonstrates the advantages of taking a long-term perspective for PB.

Example 1. *Imagine a city with five inhabitants (1 to 5). There are two districts: 1, 2 and 3 live in the first district (type t_1) and 4 and 5 in the second one (type t_2). A PB process will be run over the next three years with a vote occurring every year. The following table indicates—per year—the proposed projects, their cost and the agents' approval ballots. A check mark (✓) indicates that the agent approves of the project. The budget limit for every year is* 10.

		Year 1				Year 2				Year 3			
Projects		$\boxed{p_1}$	p_2	p_3	$\boxed{p_4}$	p_5	$\boxed{p_6}$	$\boxed{p_7}$	$\boxed{p_8}$	$\boxed{p_9}$	p_{10}	$\boxed{p_{11}}$	p_{12}
Cost		6	2	2	4	5	5	3	2	7	7	4/3	29/3
t_1	**1**	✓			✓		✓	✓	✓	✓		✓	
	2	✓		✓			✓		✓	✓		✓	
	3	✓			✓		✓	✓	✓	✓		✓	
t_2	**4**		✓		✓	✓		✓	✓		✓	✓	✓
	5			✓	✓	✓		✓			✓		✓

We assume an online *process: when the budgeting decision has to be made for a given year, only the past is known and agents have no knowledge about future years.*

Now, suppose that the municipality wants to select budget allocations that maximize either the total number of approvals[1] *or the total number of approvals weighted by the cost*[2] *(the most commonly used methods in real-life [Aziz and Shah, 2020]). In both cases, all projects that are boxed in the table would be selected (if ties are broken accordingly).*

It can be argued that these outcomes are not particularly fair with respects to the two districts. Agents in district t_1 are favored by the outcomes. For example, 1, 2 and 3 approve on average roughly 90% *of the selected projects, while 4 and 5 only roughly* 40%*. As will be shown later in the paper (Examples 2 and 4), it is actually possible to reach a much more equal treatment of the two districts in the last year, by taking into account what happened during the first two years. In this paper, we introduce concepts that make precise in which sense the modified solutions are fairer than the original one, and we discuss whether fair solutions are guaranteed to exist.*

3 Perpetual Participatory Budgeting

In essence, our framework consists of a sequence of budgeting problems over several rounds. Let $\mathfrak{P}$ be the set of all the projects occurring throughout the process. Their cost is given by the *cost function* $c : \mathfrak{P} \to \mathbb{N}$. To simplify the notation, we will write $c(P)$ instead of $\sum_{p \in P} c(p)$ for any $P \subseteq \mathfrak{P}$. Moreover, let $\mathcal{N}$ be the *set of agents* taking part in the process; we assume this set to remain the same in all rounds. Every agent belongs to a type that can represent the district she lives in or any other characteristics. Observe that each agent can have her own type. All our fairness notions and results extend to this special case. Let $\mathcal{T}$ be the set of types, *the type function* $T : \mathcal{N} \to \mathcal{T}$ indicates for every agent $i \in \mathcal{N}$ her type $T(i)$. For simplicity, we will sometimes consider a type $t \in \mathcal{T}$ as the set of agents having this type $\{i \in \mathcal{N} \mid T(i) = t\}$. In that respect, $|t|$ denotes the number of agents having type $t \in \mathcal{T}$.

Definition 1 (Budgeting problem)**.** *A budgeting problem for round j is defined by the tuple $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$ where:*

- *$\mathcal{P}_j \subseteq \mathfrak{P}$ is the set of available projects,*
- *$b_j \in \mathbb{N}_{>0}$ is the available budget,*
- *$A_j : \mathcal{N} \to 2^{\mathcal{P}_j}$ is the approval function giving for every $i \in \mathcal{N}$ the set of projects $A_j(i) \subseteq \mathcal{P}_j$ she approves of.*

We also make the assumption that every project is approved by at least one agent and that every agent approves of at least one project; projects without approvals as well as agents with empty ballots can be removed in a pre-processing stage.

The outcome of a budgeting problem $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$ is a budget allocation $\pi_j \subseteq \mathcal{P}_j$. It is *feasible* if $c(\pi_j) \leq b_j$ and $\mathcal{A}(I_j)$ is the set of all feasible budget allocations for I_j. It is *exhaustive* if it is feasible and there is no project $p \in \mathcal{P}_j \setminus \pi$ such that $c(\pi \cup \{p\}) \leq b_j$. We also speak about feasible and exhaustive ballots using the same definition. Feasible ballots are usually referred to as knapsack ballots.

[1]The sum of approvals of the selected projects.

[2]The sum over the selected projects of their cost times approval.

A *perpetual participatory budgeting instance* of length $k \in \mathbb{N}_{>0} \cup \{\infty\}$ (or k-PPB instance) is a sequence of k budgeting problems $\boldsymbol{I} = (I_1, \ldots, I_k)$. A vector $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ where $\pi_j \subseteq \mathcal{P}_j$ for every round $j \in \{1, \ldots, k\}$ will be called a *solution* for $\boldsymbol{I}$. It is said to be *feasible* (resp. exhaustive) for $\boldsymbol{I}$ if every $\pi_j \in \boldsymbol{\pi}$ is feasible (resp. exhaustive) for I_j.[3]

4 A Fairness Theory for PPB

Solutions can benefit some types while disadvantaging others. To be able to reason about the quality of solutions, we will introduce several *fairness criteria*. In order to discuss whether a solution is fair or unfair, we first need a way to measure the welfare of types.

Definition 2 (Welfare Measure)**.** *A* welfare measure F *is a function taking as inputs a k-PPB instance $\boldsymbol{I}$, a solution $\boldsymbol{\pi}$, a type $t \in \mathcal{T}$ and a round $j \in \{1, \ldots, k\}$ and returning the welfare score $F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) \in \mathbb{R}$ for type t of the solution $\boldsymbol{\pi}$ for the first j rounds of $\boldsymbol{I}$.*

Let us begin with fairness criteria. Specific welfare measures are introduced in a second time.

4.1 Fairness Criteria

The foundation of our fairness theory is that the fairest solution should be so that all types are treated exactly the same and thus enjoy the same level of welfare. This requirement is our first fairness criteria.

Definition 3 (Equal-F)**.** *For a welfare measure F, a solution $\boldsymbol{\pi}$ for the k-PPB instance $\boldsymbol{I}$ satisfies* equal-F *at round $j \in \{1, \ldots, k\}$ if for every two types $t, t' \in \mathcal{T}$, we have:*

$$F(\boldsymbol{I}, \boldsymbol{\pi}, t, j) = F(\boldsymbol{I}, \boldsymbol{\pi}, t', j).$$

Moreover, a solution $\boldsymbol{\pi}$ satisfies equal-F *if it is equal-F at round j for all rounds $j \in \{1, \ldots, k\}$.*

As Equal-F can be too strong of a requirement, we introduce two relaxations in the following.

A first approach when perfect fairness cannot be achieved, is to try to optimize for it. This idea is particularly relevant when the long-term perspective is adopted as subsequent rounds can compensate for unfairness in previous rounds. We pursue this approach by introducing the Gini coefficient [Gini, 1912] of a solution—a well-known measure of inequality given a multi-set of values—that can be used as a minimization objective. In the following, we use the standard formulation [Blackorby and Donaldson, 1978].

Definition 4 (F-Gini)**.** *Let $\vec{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$ be a vector ordered in non-increasing order, i.e., such that $v_i \geq v_j$ for all $1 \leq i \leq j \leq k$. The Gini coefficient of $\vec{v}$ is given by:*

$$gini(\vec{v}) = 1 - \frac{\sum_{i=1}^{k}(2i-1)v_i}{k\sum_{i=1}^{k} v_i}.$$

For a welfare measure F, the F-Gini coefficient of a solution $\boldsymbol{\pi}$ for the k-PPB instance $\boldsymbol{I}$ at round $j \in \{1, \ldots, k\}$ is then:

$$gini_F(\boldsymbol{I}, \boldsymbol{\pi}, j) = gini(\vec{F}(\boldsymbol{I}, \boldsymbol{\pi}, j)),$$

where $\vec{F}(\boldsymbol{I}, \boldsymbol{\pi}, j)$ is a vector containing $F(\boldsymbol{I}, \boldsymbol{\pi}, t, j)$ for all types $t \in \mathcal{T}$, ordered in non-increasing order.

A solution $\boldsymbol{\pi}$ is F-Gini-optimal at round j *with respect to a set S of solutions for $\boldsymbol{I}$, if there is no solution $\boldsymbol{\pi}' \in S \setminus \{\boldsymbol{\pi}\}$ with $gini_F(\boldsymbol{I}, \boldsymbol{\pi}', j) < gini_F(\boldsymbol{I}, \boldsymbol{\pi}, j)$.*

It can be checked that F-Gini-optimality is indeed a relaxation of equal-F in the sense that for all welfare measure F, a solution $\boldsymbol{\pi}$ satisfies equal-F if and only if its F-Gini coefficient reaches 0 (the minimum of the F-Gini coefficient).

Another approach we follow is to require perfect fairness but only in the long run. For that we introduce *convergence to equal-F* which formalizes the idea of asymptotically equalizing the welfare of the different types.

Definition 5 (Convergence to equal-F)**.** *For a welfare measure F, a solution $\boldsymbol{\pi}$ for the ∞-PPB instance $\boldsymbol{I}$* converges to equal-F *if for every two types $t, t' \in \mathcal{T}$:*

$$\frac{F(\boldsymbol{I}, \boldsymbol{\pi}, t, k)}{F(\boldsymbol{I}, \boldsymbol{\pi}, t', k)} \underset{k \to +\infty}{\longrightarrow} 1.$$

We will study the computational complexity of the following problems related to these fairness criteria. Note that this computational analysis does not apply for convergence to equal-F as we deal with infinite sequences there.

EQUAL-F	
Input:	A k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ and a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{k-1})$.
Question:	Is there a non-empty and feasible budget allocation π_k for I_k such that $(\pi_1, \ldots, \pi_{k-1}, \pi_k)$ provides equal-F at round k?

F-GINI-OPTIMALITY	
Input:	A k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ and a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$.
Question:	Is $\boldsymbol{\pi}$ Gini-optimal at round k w.r.t. all non-empty, feasible solutions?

4.2 Welfare Measures

The first welfare measures we define are based on the satisfaction of an agent. Even though agents can have personal utility functions to express their satisfaction for a given outcome (e.g. [Peters *et al.*, 2020]), this information is usually private, i.e., unknown to the decision maker. We thus need to approximate the satisfaction of an agent. We use a standard definition for satisfaction [Talmon and Faliszewski, 2019].

Definition 6 (Satisfaction)**.** *Let $\boldsymbol{I} = (I_1, \ldots, I_k)$ be a k-PPB instance and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ a solution for $\boldsymbol{I}$. For round $j \in \{1, \ldots, k\}$, whose budgeting problem is $\langle \mathcal{P}_j, b_j, A_j \rangle$, we define the* marginal satisfaction *of agent $i \in \mathcal{N}$ as:*

$$sat_j^m(\boldsymbol{I}, \pi_j, i) = c(\pi_j \cap A_j(i)).$$

Moreover, the marginal satisfaction *and the* satisfaction *of a type $t \in \mathcal{T}$ for round $j \in \{1, \ldots, k\}$ are defined by:*

$$sat_j^m(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} sat_j^m(\boldsymbol{I}, \pi_j, i)$$

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = \sum_{1 \leq j^* \leq j} sat_{j^*}^m(\boldsymbol{I}, \pi_{j^*}, t).$$

[3] One could weaken the feasibility requirement of solutions by allowing unused budget to be used in later rounds. For our results, it is not relevant which definition we take.

	Satisfaction				Relative Satisfaction			Share		
	2 agents	3 agents	> 3 agents	Complex.	2 types	> 2 types	Complex.	2 agents	> 2 agents	Complex.
equal-F	✗	✗	✗	NP-c	✗	✗	NP-c	✗	✗	NP-c
conv. to equal-F	✓	✓ (ex. ballots)	✗		✓ (knap. ballots)	?		✓	✗	
F-Gini optimality	✓	✓	✓	co-NP-c	✓	✓	co-NP-c.	✓	✓	co-NP-c

Table 1: Summary of the results. The columns specifying a number of agents/types are for existence guarantees: a ✓ indicates that for all instances with the specified number of agents/types, there exists a solution satisfying the fairness criteria; and the ✗ the opposite. The tags "ex. ballots" and "knap. ballots" indicates that the result only holds with exhaustive or knapsack ballots. The column "Complex." indicates the computational complexity of the problems stated in Section 4.1 where NP-c stands for NP complete and co-NP-c for co-NP complete.

Example 2. *Let us illustrate satisfaction on Example 1. It can be checked that by the end of year 2, the satisfaction of type t_1 is $17 + {}^2/_3$ while that of type t_2 is only 8. Hence, selecting only p_{12} in the last year would lead to a solution such that both types would have a satisfaction of $17 + {}^2/_3$.*

One potential drawback of satisfaction is its strong dependence on voters' approval sets. For example, if agent 1 approves a proper subset of agent 2's approved projects ($A_j(1) \subset A_j(2)$) and all their approved projects are funded ($A_j(2) \subseteq \pi_j$), then agent 2 is more satisfied than agent 1. However, it can be argued that the welfare of both agents should be equal as all projects they wanted to be funded have actually been funded; neither agent 1 or 2 can be made happier (subject to the available information). To take this into account, we define *relative satisfaction* which measures how close the satisfaction of an agent is to his best-case scenario.

Definition 7 (Relative satisfaction)**.** *Let $\boldsymbol{I} = (I_1, \dots, I_k)$ be a k-PPB instance and $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ a solution for $\boldsymbol{I}$. For round $j \in \{1, \dots, k\}$ corresponding to the budgeting problem $\langle \mathcal{P}_j, b_j, A_j \rangle$, we define the* marginal relative satisfaction *of agent $i \in \mathcal{N}$ as:*

$$rsat_j^m(\boldsymbol{I}, \pi_j, i) = \frac{c(\pi_j \cap A_j(i))}{\max\{c(A) \mid A \subseteq A_j(i) \text{ and } c(A) \leq b_j\}}.$$

Moreover, marginal relative satisfaction *and the* relative satisfaction *of a type $t \in \mathcal{T}$ for the round $j \in \{1, \dots, k\}$ are defined as follows:*

$$rsat_j^m(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} rsat_j^m(\boldsymbol{I}, \pi_j, i)$$
$$rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = \sum_{1 \leq j^* \leq j} rsat_{j^*}^m(\boldsymbol{I}, \pi_{j^*}, t).$$

Example 3. *In Example 1, the relative satisfaction scores by the end of year are ${}^{23}/_{12}$ for type t_1 and ${}^{53}/_{48}$ for type t_2. One can verify that there is no budget allocation for the third year that would lead to equal-relative satisfaction.*

Satisfaction and relative satisfaction are two concepts which relate to the idea of utilitarianism. Indeed, only the impact of the selected solution on the agents or types is taken into consideration and not the way the resources were spent. Although utilitarian welfare is attractive, other notions can be considered in participatory budgeting. The most important alternative might be distributive welfare which aims at spending an equal amount of resources on each agent or type. To account for distributive welfare, we introduce another welfare measure called the *share* of a type.

Definition 8 (Share)**.** *Let $\boldsymbol{I} = (I_1, \dots, I_k)$ be a k-PPB instance with a solution $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$. For round $j \in \{1, \dots, k\}$ with budgeting problem $\langle \mathcal{P}_j, b_j, A_j \rangle$, the* marginal share *of agent $i \in \mathcal{N}$ is defined as:*

$$share_j^m(\boldsymbol{I}, \pi_j, i) = \sum_{p \in \pi_j \cap A_j(i)} \frac{c(p)}{|\{i' \in \mathcal{N} \mid p \in A_j(i')\}|}$$

Moreover, the marginal share *and the* share *of a type $t \in \mathcal{T}$ for round $j \in \{1, \dots, k\}$ are defined as:*

$$share_j^m(\boldsymbol{I}, \pi_j, t) = \frac{1}{|t|} \sum_{i \in t} share_j^m(\boldsymbol{I}, \pi_j, i)$$
$$share_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = \sum_{1 \leq j^* \leq j} share_{j^*}^m(\boldsymbol{I}, \pi_{j^*}, t).$$

Example 4. *Once again coming back to Example 1, we can show that equal-share can be achieved by the end of the last year. Indeed by the end of year 2, the shares are $5 + {}^1/_3$ for t_1 and 2 for t_2. Now, by selecting p_{10} and p_{11} in the third year, we reach a solution where each type has a share of $5 + {}^2/_3$.*

Observe that trying to equalize the shares of the different types requires the *average* share of each type to be equal, meaning that we require the total share of a type to be proportional to its size. In this sense, the fairness criteria equal-share can be considered a proportionality concept.

Note that relative share—in contrast to relative satisfaction—is not a very sensible property as distributive fairness should hardly depend on the ballots.

5 Realizing Fairness

We will now explore the fairness criteria and the welfare measures we have defined previously. All our results are summarized in Table 1. Note that most of the proofs are omitted due to space constraints but can be found in the appendix.

5.1 Achieving Perfect Fairness: Equal-F

We first explore the criteria that we consider to represent a situation of perfect fairness: equal-F.

Unfortunately, it is easy to check that equal-F cannot be guaranteed even for a single round (except by selecting an empty budget allocation) for all of our welfare measures. Consider the following example with two agents where no non-empty solution satisfies either equal-satisfaction, equal-relative satisfaction or equal-share in any round.

Example 5. *Let $\boldsymbol{I}$ be a k-PPB instance with two agents 1 and 2 of types t_1 and t_2 respectively. Furthermore, let $b_j = 1$ for every round $j \in \{1, \dots, k\}$ and let $c(p) = 1$ for all $p \in \mathfrak{P}$.*

In the first round, agent 1 approves only of project p_1 and agent 2 only of p_2. In all following rounds, they both only approve of p_1. Assume w.l.o.g. that p_1 is selected in the first round. Then, for every solution $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$, at round $j \in \{2, \dots, k\}$, we have $F_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = 1 + F_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)$ for all three of our welfare measures F.

The example shows that in general, equal-F cannot be satisfied for our welfare measures. However, it could still be achieved on some specific instances. It turns out that for all three welfare measures, checking the existence of an equal-F solution is an NP-complete problem.

Proposition 1. *The* EQUAL-SATISFACTION *and* EQUAL-RELATIVE SATISFACTION *problems are strongly NP-complete even if there is only one round.*

Proposition 2. *The* EQUAL-SATISFACTION *and* EQUAL-SHARE *problems are weakly NP-complete even if there is only one round and there are only two agents.*

Proof (Sketch). We reduce from SUBSET SUM [Karp, 1972], i.e., the problem of finding a subset Z' of a set $Z \subset \mathbb{Z}$ such that $\sum Z' = 0$: Consider a PB instance with a project p_z for every $z \in Z$ such that $c(p_z) = |z|$. Further, there are two agents, v_+ and v_- approving all projects p_z with $z \geq 0$ resp. $z < 0$. We claim that this is a positive instance of EQUAL-SATISFACTION and EQUAL-SHARE if and only if Z is a positive instance of SUBSET SUM. □

We observe that both results still hold if we additionally require exhaustiveness, i.e., if we ask whether there is an exhaustive solution that satisfies equal-F.

5.2 Optimizing for Fairness: F-Gini-optimality

Let us now turn our attention to F-Gini-optimality. Note first that—by definition—there will always be for every instance at least one solution which is F-Gini-optimal. Therefore, the main questions here concern computational problems and are not about existence guarantees.

We first show that F-GINI-OPTIMALITY is co-NP-complete for both satisfaction and share.

Proposition 3. SATISFACTION-GINI-OPTIMALITY, RELATIVE-SATISFACTION-GINI-OPTIMALITY *and* SHARE-GINI-OPTIMALITY *are weakly co-NP-complete even if there is only one round and two agents.*

We note that it is also co-NP-complete to check whether a solution is Gini-optimal among exhaustive solutions. An interesting open question is the complexity of finding a Gini-optimal solution that maximizes the overall welfare of the population.

5.3 Achieving Fairness in the Long Run: Convergence to Equal-F

Let us conclude our analysis by investigating convergence to equal-F. We first show that for two agents convergence to equal-satisfaction can always be guaranteed (under mild additional assumptions).

Proposition 4. *Consider an ∞-PPB instance $\boldsymbol{I}$ with two agents such that there exists a constant $B^\star \in \mathbb{N}$ with $b_j \leq B^\star$ for every round j. Furthermore, assume for every round and both agents that there is a project p with $c(p) \leq b_j$ that the agent approves of. Then, there is a non-empty feasible solution that converges to equal-satisfaction.*

Proof. Call the agents 1 and 2 and assume they belong to types t_1 and t_2 respectively (as equal-satisfaction is trivially satisfied if there is only one type). We claim that there exists a solution $\boldsymbol{\pi}$ such that for every round j, we can guarantee:

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star.$$

Let us prove the claim by induction. For the first round, it is clear that whichever non-empty budget allocation has been chosen, we have $0 \leq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq B^\star$ for $t \in \{t_1, t_2\}$.

Now assume the claim holds for round $j-1$. W.l.o.g. assume $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1)$. Let p be a project approved by 2 such that $c(p) \leq b_j$. Then, we set $\pi_j = \{p\}$. This implies that

$$sat_j^m(\boldsymbol{I}, \pi_j, t_1) \leq sat_j^m(\boldsymbol{I}, \pi_j, t_2) \leq B^*$$

From this together with the induction hypothesis and the assumption $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1)$ we can conclude that the claim also holds in round j.

Now, we know that $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \geq \sum_{j'=1}^{j} c(\pi_{j'})$. Together with the claim, this implies that $\lim_{j \to +\infty}(sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)) = \lim_{j \to +\infty}(sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)) = +\infty$. Therefore, the proposition follows from the following:

$$\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \leq \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \leq \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)}.$$

□

Unfortunately, this result cannot be generalized—even for three agents—as the following example shows.

Example 6. *Let $\boldsymbol{I}$ be a ∞-PPB instance with three agents $1, 2, 3$ where agent 1 has type t_1 and agents 2 and 3 have type t_2. Assume $b_j = 1$ for every round j and $c(p) = 1$ for all projects $p \in \mathfrak{P}$. In every round, there are two projects and agent 1 approves of both, 2 approves of only one and 3 of the other one. Then, for every non-empty feasible solution $\boldsymbol{\pi}$ and every round j, we have $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = j$ and $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) = \frac{j}{2}$. Therefore, we have*

$$\lim_{j \to +\infty} \left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \right) = \frac{1}{2}.$$

This counter-example can be avoided if we impose some restrictions on the ballots the agents may submit. Indeed, if ballots are exhaustive then, for three agents we can always find a solution that converge to equal-satisfaction.

Proposition 5. *Consider an ∞-PPB instance $\boldsymbol{I}$ with three agents where the ballot of each agent is exhaustive in every round and there exists a constant $B^\star \in \mathbb{N}$ with $b_j \leq B^\star$ for every round j. Then, there is a non-empty feasible solution that converges to equal-satisfaction.*

However, by increasing the number of agents we again encounter an impossibility, even with these restricted ballots.

Example 7. *Let $\boldsymbol{I}$ be a ∞-PPB instance. In every round j, we have $b_j = 10$, there are eight agents $1, \dots, 8$ such that $1, 2, 3$ have type t_1 and $4, 5, 6, 7, 8$ have type t_2. Furthermore, there are six projects $p_1, \dots, p_6$ such that $c(p_1) = c(p_2) = c(p_3) = 5$ and $c(p_4) = c(p_5) = c(p_6) = 3$. The ballots are such that, for every round j:*

$$A_j(1) = \{p_1, p_4\} \quad A_j(2) = \{p_2, p_5\} \quad A_j(3) = \{p_3, p_6\}$$
$$A_j(4) = \{p_1, p_2\} \quad A_j(5) = \{p_1, p_3\} \quad A_j(6) = \{p_2, p_3\}$$
$$A_j(7) = \{p_4, p_5, p_6\} \quad A_j(8) = \{p_4, p_5, p_6\}$$

We leave it to reader to check that for each project the marginal satisfaction for type t_2 is higher than for type t_1. This directly implies that there can be no non-empty solution converging to equal-satisfaction.

Results about convergence to equal-share are very similar to the ones with equal-satisfaction. By a similar argument as for Proposition 4, we can show that convergence to equal-share can be achieved for two agents. Unfortunately, we cannot go far beyond this, as the following example shows.

Example 8. *Consider again the same ∞-PPB instance as in Example 7. We claim that for every project, selecting it would lead to a higher share for type t_2 than that of type t_1. For project p_1, we have $share_1(\boldsymbol{I}, \{p_1\}, t_1) = \frac{1}{3} \cdot \frac{5}{3} = \frac{5}{9}$ but $share_1(\boldsymbol{I}, \{p_1\}, t_2) = \frac{1}{5}(\frac{5}{3} + \frac{5}{3}) = \frac{2}{3}$. The case for p_2 and p_3 is similar. For p_4, we have $share_1(\boldsymbol{I}, \{p_4\}, t_1) = \frac{1}{3} \cdot \frac{3}{3} = \frac{1}{3}$ but $share_1(\boldsymbol{I}, \{p_4\}, t_2) = \frac{1}{5}(\frac{3}{3} + \frac{3}{3}) = \frac{2}{5}$. The case for projects p_5 and p_6 is similar. It follows that, in this example, we cannot have convergence to equal shares.*

Results are more positive when it comes to relative satisfaction. Indeed, we can guarantee convergence to equal-relative satisfaction when there are two types. Note that this result is much more general than Lemma 4 as types may contain an arbitrary number of agents. The proof is based on the following lemma stating that with two types, we can always favor one type over the other.

Lemma 6. *Let $\boldsymbol{I}$ be a k-PPB instance with non-empty knapsack ballots and two types t_1 and t_2. Then, in every round $j \in \{1, \dots, k\}$ there are two feasible budget allocations π_1 and π_2 such that:*

$$0 < rsat_j^m(\boldsymbol{I}, \pi_1, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi_1, t_2) \qquad \textit{and}$$
$$rsat_j^m(\boldsymbol{I}, \pi_2, t_1) \leq rsat_j^m(\boldsymbol{I}, \pi_2, t_2) > 0.$$

Thanks to this lemma, using a similar line of reasoning as in Proposition 4, we can show that for two types we can find a solution that converges to equal relative satisfaction.

Theorem 7. *Assume that $\boldsymbol{I}$ is an ∞-PPB-instance with non-empty knapsack ballots such that there are only two types and a $B^\star$ such that $b_j \leq B^\star$ for all rounds j. Then, there is a non-empty feasible solution for $\boldsymbol{I}$ that converges to equal relative satisfaction.*

It is important to mention that the proofs of Lemma 6 and that of Theorem 7 are both constructive, in the sense that they show how to compute the relevant solutions. However, this construction does not guarantee the solution to be exhaustive. To achieve this, an additional ballot restriction is necessary.

Corollary 8. *Consider an ∞-PPB instance $\boldsymbol{I}$ that satisfies all the conditions of Theorem 7. Then, there exists a non-empty feasible solution $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ for $\boldsymbol{I}$ that (i) converges to equal relative satisfaction and (ii) such that for each round j there is an agent i with $A_j(i) \subseteq \pi_j$. In particular, if all ballots are exhaustive, then every budget allocation in $\boldsymbol{\pi}$ is exhaustive.*

Whether Theorem 7 and Corollary 8 can be extended to three and more types remains an important open question.

6 Conclusion

In this paper, we have introduced a model of participatory budgeting (called *perpetual participatory budgeting*) that takes into account the temporal component of a PB process. We have further defined a theory of fairness for this model by introducing several fairness criteria. We considered both egalitarian concepts based on voters' (relative) satisfaction and a form of proportionality based on shares. For corresponding axiomatic properties, we studied whether (and when) we can guarantee these to hold as well as the computational complexity of verifying them.

We can conclude that taking the long-term viewpoint allows us to approximate forms of fairness that cannot be obtained in single-round PB instances. This was already visible in our starting example in Section 2. Beyond that, we have established three strong fairness concepts that required a thorough analysis to judge their applicability. On the one hand, we have seen that they cannot be guaranteed in general. On the other hand, and we have identified special cases where some of these strong are guaranteed to hold.

Several research directions can be pursued within our proposed framework. For instance, it would be interesting to look for natural PB procedures that compute solutions satisfying (or approximating) our fairness criteria. In the light recent works (e.g., [Hershkowitz *et al.*, 2021]), a relevant question concerns the price of fairness, i.e., how much the satisfaction has to be reduced in order to achieve fairness. The fairness criteria we introduced may not be compatible with efficiency notions such as Pareto-optimality. Although the tension between fairness and efficiency is well-known (see, e.g., [Peters *et al.*, 2020]), it would be interesting to study the combination of fairness and efficiency criteria in our setting. Moreover, PB generalizes multi-winner voting in a way that makes it closer to the fair division of indivisible items [Bouveret *et al.*, 2016]. This linked was already hinted in Example 5 as applying "up-to-one project" criteria would have made the solution fair. It would then be interesting to investigate if and how fairness criteria from fair division (for instance envy-freeness or EF1) could be successfully adapted and applied to PB. Finally, by considering real-world data of repeated PB referenda one can analyze the possible gains of taking a long-term perspective as we propose here.

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A Proofs

For the proofs of Propositions 1 and 2, the reductions are done for a single round hence no solution is given in the input.

Proof of Proposition 1

Proof. Membership in NP is clear, the certificate being the solution itself. First, we will prove the hardness of EQUAL-SATISFACTION. We will reduce from the following strongly NP-hard problem [Garey and Johnson, 1979; Schaefer, 1978].

ONE-IN-THREE 3-SAT	
Input:	A propositional formula φ in conjunctive normal form with exactly three literals per clause (3-CNF).
Question:	Is there a truth assignment α for φ so that each clause in φ has exactly one literal set to true?

Consider a 3-CNF formula φ. Denote by $\mathcal{X}$ the set of propositional variables appearing in φ and by $\mathcal{C}$ the set of clauses of φ. We construct a 1-PPB instance $\boldsymbol{I} = (I_1)$ where $I_1 = \langle \mathcal{P}, b, A \rangle$ as follows. The set of projects is $\mathcal{P} = \bigcup_{x \in \mathcal{X}} \{p_x, p_{\neg x}\}$, they all have cost 1. For each propositional variable $x \in \mathcal{X}$, there is an agent i_x approving of both p_x and $p_{\neg x}$. Moreover, for each clause $c \in \mathcal{C}$, there is an agent i_c approving of the three projects corresponding to the literals in c. Every agent belongs to a unique type and is the only one belonging to that type. Finally, the budget limit is $b = |\mathcal{X}|$. This reduction can clearly be done in polynomial.

First, we show that there exists a truth assignment for φ that sets exactly one literal to true in every clause of φ if there exists an non-empty and feasible solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ that provides equal-satisfaction. Indeed, since $\boldsymbol{\pi}$ has to be non-empty, at least one agent i_x will have satisfaction 1. From equal-satisfaction, this implies that every agent should have at least satisfaction 1. Note that the approval ballots of the agents in $\{i_x \mid x \in \mathcal{X}\}$ are all disjoint. Since the budget limit is $b = |\mathcal{X}|$, equal satisfaction in $\boldsymbol{I}$ is equivalent to every agent having satisfaction 1. Moreover, reaching satisfaction 1 for every agent is equivalent to selecting exactly one project among p_x and $p_{\neg x}$ for every $x \in \mathcal{X}$, and exactly one project among the one corresponding to the literals in c for all $c \in \mathcal{C}$. Call π such a budget allocation. Observe that for π to be feasible, it should be the case that the projects selected for the "clause agents" should be the same as the ones selected for the "variable agents".[4] Consider then the truth assignment T that sets a propositional variable $x \in \mathcal{X}$ to true (resp. false) if and only if p_x (resp. $p_{\neg x}$) has been selected in π. By construction T is a suitable truth assignment for the ONE-IN-THREE 3-SAT problem.

Now, we show that if there exists a truth assignment T for φ that sets exactly one literal to true in every clause of φ then there exists an non-empty and feasible solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ that provides equal-satisfaction. Let

$$\pi = \{p_x \mid x \in T\} \cup \{p_{\neg x} \mid x \notin T\}.$$

As T is a truth assignment each variable is either true or false, hence we have $c(\pi) = |\mathcal{X}|$ and each "variable agent" has satisfaction 1. Moreover, exactly one literal must be set to true in every clause, therefore each "clause agent" has satisfaction 1. Hence, π is an exhaustive allocation that satisfies equal-satisfaction.

The reduction for EQUAL-RELATIVE SATISFACTION is similar, but we need to add an additional project with cost $|\mathcal{X}|$ that is approved by all "variable voters".

Membership in NP is clear, the certificate being the solution itself. We show hardness via a reduction from ONE-IN-THREE-3-SAT. Consider a 3-CNF formula φ.

Denote by $\mathcal{X}$ the set of propositional variables appearing in φ and by $\mathcal{C}$ the set of clauses of φ. We construct a 1-PPB instance $\boldsymbol{I} = (I_1)$ where $I_1 = \langle \mathcal{P}, b, A \rangle$ as follows. The set of projects is $\mathcal{P} = \bigcup_{x \in \mathcal{X}} \{p_x, p_{\neg x}\} \cup \{p^*\}$. All projects except p^* have cost 1, p^* has cost $|\mathcal{X}|$. For each propositional variable $x \in \mathcal{X}$, there is an agent i_x approving of p^* and of both p_x and $p_{\neg x}$. Moreover, for each clause $c \in \mathcal{C}$, there is an agent i_c approving of the three projects corresponding to the literals in c. Every agent belongs to a unique type and is the only one belonging to that type. Finally, the budget limit is $b = |\mathcal{X}|$. This reduction can clearly be done in polynomial.

First, we show that there exists a truth assignment for φ that sets exactly one literal to true in every clause of φ if there exists an non-empty and feasible solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ that provides equal relative satisfaction. Indeed, since $\boldsymbol{\pi}$ has to be non-empty, at least one agent i_x will have relative satisfaction at least $1/3$. Because of equal relative satisfaction, this implies that every agent should have at least relative satisfaction $1/3$. We observe that this implies that p^* is not in $\boldsymbol{\pi}$ as $\{p^*\}$ is the only feasible solution that contains p^* and all "clause agents" have 0 relative satisfaction if $\{p^*\}$ is selected. Note that the approval ballots of the agents in $\{i_x \mid x \in \mathcal{X}\}$ are all disjoint. Since the budget limit is $b = |\mathcal{X}|$, equal relative satisfaction in $\boldsymbol{I}$ is equivalent to every agent having relative satisfaction $1/3$. Moreover, reaching satisfaction $1/3$ for every agent is equivalent to selecting exactly one project among p_x and $p_{\neg x}$ for every $x \in \mathcal{X}$, and exactly one project among the one corresponding to the literals in c for all $c \in \mathcal{C}$. Call π such a budget allocation. Observe that for π to be feasible, it should be the case that the projects selected for the "clause agents" should be the same as the ones selected for the "variable agents". [5] Consider then the truth assignment T that sets a propositional variable $x \in \mathcal{X}$ to true (resp. false) if and only if p_x (resp. $p_{\neg x}$) has been selected in π. It is clear that T is a suitable truth assignment for the ONE-IN-THREE 3-SAT problem.

Now, we show that if there exists a truth assignment T for φ that sets exactly one literal to true in every clause of φ then there exists an non-empty and feasible solution $\boldsymbol{\pi}$ for $\boldsymbol{I}$ that provides equal relative satisfaction. Let

$$\boldsymbol{\pi} = \{p_x \mid x \in T\} \cup \{p_{\neg x} \mid x \notin T\}.$$

As T is a truth assignment each variable is either true or false, hence we have $c(\boldsymbol{\pi}) = |\mathcal{X}|$ and each "variable agent" has relative satisfaction $1/3$. Moreover, exactly one literal must be

[4] Note that since $c(\pi) = b$, π is trivially exhaustive. This will be used to prove Corollary 9.

[5] Note that since $c(\pi) = b$, π is trivially exhaustive. This will be used to prove Corollary 9.

set to true in every clause, therefore each "clause agent" has relative satisfaction 1/3. Hence, $\boldsymbol{\pi}$ is an exhaustive allocation that satisfies equal-relative-satisfaction. □

Proof of Proposition 2

Proof. We show the NP-hardness of EQUAL-SATISFACTION via a reduction from the following (weakly) NP-hard problem [Karp, 1972; Garey and Johnson, 1979]

SUBSET SUM	
Input:	A finite set $Z \subseteq (\mathbb{Z} \setminus \{0\})$.
Question:	Is there a non-empty $Z' \subseteq Z$ such that $\sum_{z \in Z'} z = 0$?

Given a set $Z \subset \mathbb{Z}$, we construct the following 1-PPB instance $\boldsymbol{I} = (I_1)$ where $I_1 = \langle \mathcal{P}, b, A \rangle$ with two agents 1 and 2, both belonging to a different type $1 \in t_+$ and $2 \in t_-$. For every $z \in Z$, there is a corresponding project $p_z \in \mathcal{P}$ with cost $|z|$. Agent 1 approves of p_z if and only $z > 0$. Agent 2 approves of p_z if and only $z < 0$. The budget limit is $b = \sum_{z \in Z} z$.

Then, we claim that for every $Z_1 \subset Z$ we have $\sum_{z \in Z_1} z = 0$ if and only if the solution defined by $\boldsymbol{\pi} = \{p_z \mid z \in Z_1\}$ satisfies equal-satisfaction. Indeed we have $\sum_{z \in Z_1} z = 0$ if and only if

$$sat_1(I, \boldsymbol{\pi}, t_+) = \sum_{\substack{z \in Z_1 \\ z > 0}} z = \sum_{\substack{z \in Z_1 \\ z < 0}} z = sat_1(I, \boldsymbol{\pi}, t_-)$$

Therefore, there is a $Z_1 \subset Z$ with $\sum_{z \in Z_1} z = 0$ if and only if there is a $\boldsymbol{\pi}$ that satisfies equal-satisfaction.

We observe that in the reduction above the ballots of both voters are disjoint. Therefore, share and satisfaction coincide in the instance. Hence, the same reduction also shows that EQUAL-SHARE is NP-complete even if we have only two voters. □

Proof that EQUAL-F stays hard with an additional exhaustiveness condition

EXHAUSTIVE EQUAL-F	
Input:	A k-PPB instance $\boldsymbol{I} = (I_1, \ldots, I_k)$ and a solution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_{k-1})$.
Question:	Is there an exhaustive and feasible budget allocation π_k for I_k such that $(\pi_1, \ldots, \pi_{k-1}, \pi_k)$ provides equal-F at round k?

Corollary 9. *The* EXHAUSTIVE EQUAL-SATISFACTION *and* EXHAUSTIVE EQUAL-RELATIVE SATISFACTION *problems are strongly NP-complete even if there is only one round.*

Proof. We observe that in the reductions in the proof of Proposition 1 any solution that satisfies equal-satisfaction or equal-relative satisfaction must also be exhaustive. Hence the reductions also directly prove that hardness of EXHAUSTIVE EQUAL-SATISFACTION and EXHAUSTIVE EQUAL-RELATIVE SATISFACTION. □

Proposition 10. *The* EXHAUSTIVE EQUAL-SATISFACTION *and* EXHAUSTIVE EQUAL-SHARE *problems are weakly NP-complete even if there is only one round and there are two agents.*

It is straightforward to adapt the proof of Proposition 2:

Proof. We show the NP-hardness of EXHAUSTIVE EQUAL-SATISFACTION and EXHAUSTIVE EQUAL-SHARE at the same time via a reduction from SUBSET SUM. Given a set $Z \subset \mathbb{Z}$, we construct the following 1-PPB instance $\boldsymbol{I} = (I_1)$ where $I_1 = \langle \mathcal{P}, b, A \rangle$ with two agents 1 and 2, both belonging to a different type $1 \in t_+$ and $2 \in t_-$. The budget limit is $b = \sum_{z \in Z} z$. For every $z \in Z$, there is a corresponding project $p_z \in \mathcal{P}$ with cost $|z|$. Agent 1 approves of p_z if and only $z > 0$. Agent 2 approves of p_z if and only $z < 0$. Furthermore, there are $b - \min\{|z| \mid z \in Z\}$ projects $p_1^*, \ldots, p^*_{b - \min\{|z| | z \in Z\}}$ with cost 1 that are approved by both voters.

First, assume there exists a non-empty $Z' \subseteq Z$ such that $\sum_{z \in Z'} z = 0$ and let

$$\boldsymbol{\pi} = \{p_z \mid z \in Z'\} \cup \{p_1^*, \ldots, p^*_{b - \sum_{z \in Z'} |z|}\}$$

Then, $\boldsymbol{\pi}$ is exhaustive by construction and we have

$$\begin{aligned} sat_1(I, \boldsymbol{\pi}, t_+) &= (\sum_{\substack{z \in Z' \\ z > 0}} z) + (b - \sum_{z \in Z'} |z|) \\ &= (\sum_{\substack{z \in Z' \\ z < 0}} z) + (b - \sum_{z \in Z'} |z|) = sat_1(I, \boldsymbol{\pi}, t_-) \quad (1) \end{aligned}$$

and

$$\begin{aligned} share_1(I, \boldsymbol{\pi}, t_+) &= (\sum_{\substack{z \in Z' \\ z > 0}} z) + \frac{1}{2}(b - \sum_{z \in Z'} |z|) \\ &= (\sum_{\substack{z \in Z' \\ z < 0}} z) + \frac{1}{2}(b - \sum_{z \in Z'} |z|) = share_1(I, \boldsymbol{\pi}, t_-) \quad (2) \end{aligned}$$

Hence, $\boldsymbol{\pi}$ is an exhaustive allocation that satisfies equal-satisfaction and equal-share.

Now assume $\boldsymbol{\pi}$ is an exhaustive allocation that satisfies equal-satisfaction or equal-share. Let $Z' = \{z \mid p_z \in \boldsymbol{\pi}\}$. Z' must be non-empty because $\boldsymbol{\pi}$ is exhaustive and, by construction, any exhaustive solution must contain at least one project of the form p_z. Furthermore, it follows for equation (1) resp. (2) that $\sum_{z \in Z'} z = 0$.

□

Proof of Proposition 3

Proof. We show that SATISFACTION-GINI-OPTIMALITY is co-NP-complete by showing that its co-problem, i.e., checking whether a solution is not Gini-optimal is NP-complete. It is clear that this problem is in NP, as we can just guess a non-empty, feasible solution and check if it has a lower Gini coefficient than $\boldsymbol{\pi}$ in polynomial time.

We show that the co-problem is NP-hard by a reduction from SUBSET-SUM. Let $Z = \{z_1, \ldots, z_k\}$ be a SUBSET-SUM instance. We construct a CO-SATISFACTION-GINI-OPTIMALITY instance as follows: There are two agents 1 and

2, with the type t_+ and t_- respectively. We only have one round. For every element $z_i \in Z$ there is a project p_i with $c(p_i) = 4|z_i|$. Furthermore, there are two projects p_+ and p_- with $c(p_+) = 8\sum_{i\leq k}|z_i|$ and $c(p_-) = 8\sum_{i\leq k}|z_i| + 1$. We have $P = \{p_1, \dots, p_k, p_+, p_-\}$. The budget is $b = \sum_{p\in P} c(p)$, i.e., in principle all projects could be funded. The approvals are given by

$$A_1 = \{p_i \mid z_i \geq 0\} \cup \{p_+\}$$

and

$$A_2 = \{p_i \mid z_i < 0\} \cup \{p_-\}.$$

Finally, we ask whether the feasible allocation $\{p_+, p_-\}$ is Gini-optimal.

We claim that there is a non-empty, feasible allocation that Gini-dominates $\{p_+, p_-\}$ if and only if Z is a positive instance of SUBSET-SUM.

First assume Z is a positive instance of SUBSET-SUM and let $Z' \subseteq Z$ be a set such that $\sum_{z\in Z'} z = 0$. We claim that the solution $\boldsymbol{\pi} = \{p_i \mid z_i \in Z'\}$ Gini-dominates $\{p_+, p_-\}$. On the one hand, we have

$$gini(\{p_+, p_-\}) = 1 - \frac{3c(p_+) + c(p_-)}{4\frac{c(p_+)+c(p_-)}{2}} = 1 - \frac{3c(p_+) + c(p_-)}{2c(p_+) + 2c(p_-)} = 1 - \frac{24\sum_{i\leq k}|z_i| + 1}{24\sum_{i\leq k}|z_i| + 2}$$

which is clearly larger than 0 because. On the other hand

$$\sum_{p_i\in A_1\cap\boldsymbol{\pi}} c(p_i) - \sum_{p_i\in A_2\cap\boldsymbol{\pi}} c(p_i) = 4\sum_{z\in Z'} s = 0$$

and hence $gini(\boldsymbol{\pi}) = 0$.

Now assume there is a non-empty, feasible allocation $\boldsymbol{\pi}$ that Gini-dominates $\{p_+, p_-\}$. We claim that

$$\sum_{p_i\in A_1\cap\boldsymbol{\pi}} c(p_i) = \sum_{p_i\in A_2\cap\boldsymbol{\pi}} c(p_i)$$

Assume otherwise there is no such solution. By construction, if the difference in share between 1 and 2 is not 0 then it must be 3, as the difference without p_+ and p_- must be a multiple of 4. In the best case, this difference can be achieved with a solution $\boldsymbol{\pi}$ that funds all projects. In that case, we would have:

$$\sum_{p_i\in A_1\cap\boldsymbol{\pi}} c(p_i) = 2\sum_{i\leq k}|z_i| + 2 + c(p_+)$$

$$\sum_{p_i\in A_2\cap\boldsymbol{\pi}} c(p_i) = 2\sum_{i\leq k}|z_i| - 2 + c(p_-)$$

Then,

$$gini(\boldsymbol{\pi}) = 1 - \frac{3(2\sum_{i\leq k}|z_i| - 2 + c(p_-)) + 2\sum_{i\leq k}|z_i| + 2 + c(p_+)}{2(2\sum_{i\leq k}|z_i| - 2 + c(p_-) + 2\sum_{i\leq k}|z_i| + 2 + c(p_+))} = 1 - \frac{30\sum_{i\leq k}|z_i| - 3 + 10\sum_{i\leq k}|z_i| + 2}{20\sum_{i\leq k}|z_i| - 2 + 20\sum_{i\leq k}|z_i| + 4)} = 1 - \frac{40\sum_{i\leq k}|z_i| - 1}{40\sum_{i\leq k}|z_i| + 2}$$

Now, to compare the two, we compute

$$\frac{24\sum_{i\leq k}|z_i| + 1}{24\sum_{i\leq k}|z_i| + 2} - \frac{40\sum_{i\leq k}|z_i| - 1}{40\sum_{i\leq k}|z_i| + 2} = \frac{32\sum_{i\leq k}|z_i| + 4}{(24\sum_{i\leq k}|z_i| + 2)(40\sum_{i\leq k}|z_i| - 2)} > 0$$

Therefore, $gini(\boldsymbol{\pi}) > gini(\{p_+, p_-\})$, a contradiction.

This means if there is a solution $\boldsymbol{\pi}$ that Gini-dominates $\{p_+, p_-\}$, then

$$\sum_{p_i\in A_1\cap\boldsymbol{\pi}} c(p_i) = \sum_{p_i\in A_2\cap\boldsymbol{\pi}} c(p_i).$$

This means $p_+, p_- \notin \boldsymbol{\pi}$, which implies that $Z' := \{z_i \mid p_i \in \boldsymbol{\pi}\}$ is a solution of the SUBSET-SUM instance.

We observe that in the reduction above the ballots of both voters are disjoint. Therefore, share and satisfaction coincide in the instance. Hence, the same reduction also shows that SHARE-GINI-OPTIMALITY is co-NP-complete even if we have only two voters.

In order to show that RELATIVE-SATISFACTION-GINI-OPTIMALITY is co-NP complete we have to modify the reduction slightly. We have to add two additional project p_1^* and p_2^* such that $c(p_1^*) = c(p_2^*) = b$ (the budget is still $b = \sum_{p\in P\setminus\{p_1^*, p_2^*\}} c(p)$). We assume that agent 1 approves project p_1^* and agent 2 approves project p_1^*. We observe that then the following holds:

$$rsat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_+) = \frac{sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_+)}{\max\{c(A) \mid A \subseteq A_1(1) \text{ and } c(A) \leq b\}} = \frac{sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_+)}{c(p_1^*)} = \frac{sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t_+)}{b}.$$

The same holds for t_-. This implies that any bundle that does not contain p_1^* and p_2^* is relative-satisfaction-Gini-optimal in the new instance if and only if the bundle is satisfaction-Gini-optimal in the original instance. We claim that, furthermore, a bundle that contains either p_1^* or p_2^* can not be relative-satisfaction-Gini-optimal. By construction, the only feasible bundle that contains p_1^* (resp. p_2^*) is $\{p_1^*\}$ (resp. $\{p_2^*\}$). Hence, the modified reduction shows that RELATIVE-SATISFACTION-GINI-OPTIMALITY is co-NP-complete. □

Proof that F-GINI-OPTIMALITY stays hard with an additional exhaustiveness condition

Exhaustive F-GINI-OPTIMALITY	
Input:	A k-PPB instance $\boldsymbol{I} = (I_1, \dots, I_k)$ and an exhaustive solution $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$.
Question:	Is $\boldsymbol{\pi}$ Gini-optimal in round k w.r.t. all exhaustive, feasible solutions?

Proposition 11. EXHAUSTIVE SATISFACTION-GINI-OPTIMALITY *and* EXHAUSTIVE RELATIVE-SATISFACTION-GINI-OPTIMALITY *are weakly co-NP-complete even if there is only one round and two agents.*

Proof. We show that the problem is co-NP-complete by showing that its co-problem, i.e., checking whether a solution is not Gini-optimal in round k is NP-complete. This problem is in NP, as we can guess an exhaustive solution and check if it has a smaller Gini coefficient in round k than $\boldsymbol{\pi}$ in polynomial time. In the following, we will say that a solution $\boldsymbol{\pi}$ Gini-dominates another solution $\boldsymbol{\pi}'$ if it has a lower Gini coefficient in round k.

We show NP-hardness via a reduction from SUBSET SUM. Consider an instance of the SUBSET SUM problem $Z \subseteq \mathbb{Z}$ and let $\zeta = \sum_{z \in Z} |z|$. We will now construct an instance of EXHAUSTIVE SATISFACTION-GINI-OPTIMALITY. First, consider the 1-PPB instance $\boldsymbol{I} = (I_1)$ where $I_1 = \langle \mathcal{P}, b, A \rangle$. There are two agents 1 and 2 of type t_+ and t_- respectively. For every $z \in Z$ there is a project p_z with $c(p_z) = 4|z|$. There is a additional project p_+ with $c(p_+) = 1$. Let $P = \{p_z \mid z \in Z\} \cup \{p_+, p_-\}$. The budget limit is then $b = 4\zeta$. There are then $b - 1$ additional projects $P^* = \{p_1^*, \dots, p_{b-1}^*\}$, all of cost 1. The ballots are given by

$$A_1(1) = \{p_z \mid z \in Z \text{ and } z > 0\} \cup \{p_+\} \cup P^*$$
$$A_1(2) = \{p_z \mid z \in Z \text{ and } z < 0\} \cup P^*.$$

Finally, the second part of the EXHAUSTIVE SATISFACTION-GINI-OPTIMALITY instance, the solution $\boldsymbol{\pi}$, consists of p^+ together with all projects in P^*, i.e.,

$$\boldsymbol{\pi} = \{p_+\} \cup P^*.$$

Clearly $c(\boldsymbol{\pi}) = b$, hence $\boldsymbol{\pi}$ is exhaustive. Further, observe that

$$sat(\boldsymbol{I}, \boldsymbol{\pi}, t_+) = b,$$
$$sat(\boldsymbol{I}, \boldsymbol{\pi}, t_-) = b - 1.$$

We claim that there is an exhaustive solution that Gini-dominates $\boldsymbol{\pi}$ if and only if Z is a positive instance of the SUBSET SUM problem. First assume Z is indeed a positive instance. Let then $Z' \subseteq Z$ be a set such that $\sum_{z \in Z'} z = 0$ and let $P' = \{p_z \mid z \in Z'\}$. We claim that the solution $\boldsymbol{\pi}' = (\pi')$ where

$$\pi' = P' \cup \left\{p_1^*, \dots p_{b-c(P')}^*\right\}$$

is exhaustive and Gini-dominates $\boldsymbol{\pi}$. As $c(\pi') = b$, $\boldsymbol{\pi}'$ clearly is exhaustive. Now, to show that $\boldsymbol{\pi}'$ Gini-dominates $\boldsymbol{\pi}$ consider first

$$gini(\boldsymbol{\pi}) = 1 - \frac{3(b-1) + b}{2(b - 1 + b)} = 1 - \frac{4b - 3}{4b - 2}$$

which is clearly larger than 0. On the other hand,

$$sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) = \sum_{p_i \in A_1(1) \cap \pi' \setminus P^*} c(p_i) + \sum_{p_i^* \in \pi' \cap P^*} c(p_i^*)$$
$$= \sum_{p_i \in A_2(2) \cap \pi' \setminus P^*} c(p_i) + \sum_{p_i^* \in \pi' \cap P^*} c(p_i^*) = sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-).$$

and hence $gini(\boldsymbol{\pi}') = 0$.

Now assume there is an exhaustive solution $\boldsymbol{\pi}' = (\pi')$ that Gini-dominates $\boldsymbol{\pi}$. First of all, observe that any exhaustive allocation must contain at least one project from $\mathcal{P} \setminus P^*$, because, by construction P^* is not exhaustive as we could add p^+. Furthermore, by construction

$$sat_1(\boldsymbol{I}, \pi' \cap P^*, t_+) = \sum_{p_i^* \in \pi' \cap P^*} c(p_i^*) = sat_1(\boldsymbol{I}, \pi' \cap P^*, t_-). \tag{3}$$

First, assume that the following holds

$$sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) = sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-). \tag{4}$$

We claim that π' cannot contain p_+. If π' contains p_+ $sat_1(\boldsymbol{I}, \pi' \setminus P^*, t_+)$ is odd, while $sat_1(\boldsymbol{I}, \pi' \setminus P^*, t_-)$ is even. Together with (3) this contradicts assumption (4). Therefore, π' can only contain projects in $\{p_z \mid z \in Z\} \cup P^*$. Hence, it follows from (3) and (4) that

$$\sum_{p \in \pi' \cap P \cap A_1(1)} c(p) = \sum_{p \in \pi' \cap P \cap A_1(2)} c(p).$$

Hence, $\sum_{z \in Z'} z = 0$ for $Z' = \{z \in Z \mid p_z \in \pi'\}$ and, by the above, we know that $Z' \neq \emptyset$. Z is thus a positive instance of SUBSET SUM.

Now, assume

$$sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) \neq sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-).$$

In the best case, $\boldsymbol{\pi}'$ can have

$$sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) = b \quad \text{and} \quad sat_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-) = b - 1.$$

as any other distribution of satisfaction must have a bigger difference in satisfaction or a lower sum of satisfaction scores. In both cases, the Gini coefficient would increase. This is the same distribution of satisfaction as for $\boldsymbol{\pi}$, hence $gini(\boldsymbol{\pi}') = gini(\boldsymbol{\pi})$ and $\boldsymbol{\pi}'$ does not Gini-dominate $\boldsymbol{\pi}$.

Finally, we observe that for both agents there is a feasible solution that gives them satisfaction b. Therefore, for $t \in \{t_+, t_-\}$ it holds that

$$rsat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) = \frac{sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t)}{b}.$$

This implies that a bundle is relative-satisfaction-Gini-optimal if and only if the bundle is satisfaction-Gini-optimal. Hence, the reduction also shows that EXHAUSTIVE-RELATIVE-SATISFACTION-GINI-OPTIMALITY is co-NP-complete. □

Essentially the same proof also works for EXHAUSTIVE SHARE-GINI-OPTIMALITY but some computations are different.

Proposition 12. EXHAUSTIVE SHARE-GINI-OPTIMALITY *is weakly co-NP-complete even if there is only one round and two agents.*

Proof. We show that the problem is co-NP-complete by showing that its co-problem, i.e., checking whether a solution is not Gini-optimal is NP-complete. This problem is in NP, as we can guess an exhaustive solution and check if it Gini-dominates $\boldsymbol{\pi}$ in polynomial time.

We will show NP-hardness via a reduction from SUBSET SUM. Consider an instance of the SUBSET SUM problem $Z \subseteq$

$\mathbb{Z}$ and let $\zeta = \sum_{z \in Z} |z|$. We will now construct an instance of EXHAUSTIVE SHARE-GINI-OPTIMALITY. First, consider the 1-PPB instance $\boldsymbol{I} = (I_1)$ where $I_1 = \langle \mathcal{P}, b, A \rangle$. There are two agents 1 and 2 of type t_+ and t_- respectively. For every $z \in Z$ there is a project p_z with $c(p_z) = 4|z|$. There is a additional project p_+ with $c(p_+) = 1$. Let $P = \{p_z \mid z \in Z\} \cup \{p_+, p_-\}$. The budget limit is then $b = 4\zeta$. There are then $b - 1$ additional projects $P^* = \{p_1^*, \ldots, p_{b-1}^*\}$, all of cost 1. The ballots are given by

$$A_1(1) = \{p_z \mid z \in Z \text{ and } z > 0\} \cup \{p_+\} \cup P^*$$
$$A_1(2) = \{p_z \mid z \in Z \text{ and } z < 0\} \cup P^*.$$

Finally, the second part of the EXHAUSTIVE SHARE-GINI-OPTIMALITY instance, the solution $\boldsymbol{\pi}$, consists of p^+ together with all projects in P^*, i.e.,

$$\boldsymbol{\pi} = \{p_+\} \cup P^*.$$

Clearly $c(\boldsymbol{\pi}) = b$, hence $\boldsymbol{\pi}$ is exhaustive. Further, observe that

$$share_1(\boldsymbol{I}, \boldsymbol{\pi}, t_+) = 1 + \frac{1}{2}(b-1) = \frac{1}{2}b + \frac{1}{2},$$
$$share_1(\boldsymbol{I}, \boldsymbol{\pi}, t_-) = \frac{1}{2}b - \frac{1}{2}.$$

We claim that there is an exhaustive solution that Gini-dominates $\boldsymbol{\pi}$ if and only if Z is a positive instance of the SUBSET SUM problem. First assume Z is indeed a positive instance. Let then $Z' \subseteq Z$ be a set such that $\sum_{z \in Z'} z = 0$ and let $P' = \{p_z \mid z \in Z'\}$. We claim that the solution $\boldsymbol{\pi}' = (\pi')$ where

$$\pi' = P' \cup \left\{p_1^*, \ldots p_{b-c(P')}^*\right\}$$

is exhaustive and Gini-dominates $\boldsymbol{\pi}$. As $c(\pi') = b$, $\boldsymbol{\pi}'$ clearly is exhaustive. Now, to show that $\boldsymbol{\pi}'$ Gini-dominates $\boldsymbol{\pi}$ consider first

$$gini(\boldsymbol{\pi}) = 1 - \frac{3(\frac{1}{2}b - \frac{1}{2}) + \frac{1}{2}b + \frac{1}{2}}{2(\frac{1}{2}b - \frac{1}{2} + \frac{1}{2}b + \frac{1}{2})} = 1 - \frac{2b-1}{2b}$$

which is clearly larger than 0. On the other hand,

$$\begin{aligned} &share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) = \\ &\sum_{p_i \in A_1(1) \cap \pi' \setminus P^*} c(p_i) + \frac{1}{2} \sum_{p_i^* \in \pi' \cap P^*} c(p_i^*) = \\ &\sum_{p_i \in A_2(2) \cap \pi' \setminus P^*} c(p_i) + \frac{1}{2} \sum_{p_i^* \in \pi' \cap P^*} c(p_i^*) = \\ &share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-). \end{aligned}$$

and hence $gini(\boldsymbol{\pi}') = 0$.

Now assume there is an exhaustive solution $\boldsymbol{\pi}' = (\pi')$ that Gini-dominates $\boldsymbol{\pi}$. First of all, observe that any exhaustive allocation must contain at least one project from $\mathcal{P} \setminus P^*$, because, by construction P^* is not exhaustive as we could add p^+. Furthermore, by construction

$$share_1(\boldsymbol{I}, \pi' \cap P^*, t_+) = \frac{1}{2} \sum_{p_i^* \in \pi' \cap P^*} c(p_i^*) = share_1(\boldsymbol{I}, \pi' \cap P^*, t_-). \quad (5)$$

First, assume that the following holds

$$share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) = share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-). \quad (6)$$

We claim that π' cannot contain p_+. If π' contains p_+ $share_1(\boldsymbol{I}, \pi' \setminus P^*, t_+)$ is odd, while $share_1(\boldsymbol{I}, \pi' \setminus P^*, t_-)$ is even. Together with (3) this contradicts assumption (4). Therefore, π' can only contain projects in $\{p_z \mid z \in Z\} \cup P^*$. Hence, it follows from (5) and (6) that

$$\sum_{p \in \pi' \cap P \cap A_1(1)} c(p) = \sum_{p \in \pi' \cap P \cap A_1(2)} c(p).$$

Hence, $\sum_{z \in Z'} z = 0$ for $Z' = \{z \in Z \mid p_z \in \pi'\}$ and, by the above, we know that $Z' \neq \emptyset$. Z is thus a positive instance of SUBSET SUM.

Now, assume

$$share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) \neq share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-).$$

Then $|share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) - share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-)|$ must be at least one, as there is no project with cost less than one. Further, we know $share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) + share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-) = b$. Hence, the best case for $\boldsymbol{\pi}'$ is

$$share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_+) = \frac{1}{2}b + \frac{1}{2}$$

and

$$share_1(\boldsymbol{I}, \boldsymbol{\pi}', t_-) = \frac{1}{2}b - \frac{1}{2}.$$

This is the same distribution of satisfaction as for $\boldsymbol{\pi}$, hence $gini(\boldsymbol{\pi}') = gini(\boldsymbol{\pi})$ and $\boldsymbol{\pi}'$ does not Gini-dominate $\boldsymbol{\pi}$. □

Proof of Proposition 4

Proof. Call the agents 1 and 2 and assume they belong to types t_1 and t_2 respectively (as equal-satisfaction is trivially satisfied if there is only one type). We claim that there exists a solution $\boldsymbol{\pi}$ such that for every round j, we can guarantee:

$$\begin{aligned} sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star \leq\ & sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \\ &\leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star. \end{aligned}$$

Let us prove the claim by induction. For the first round, it is clear that whichever non-empty budget allocation has been chosen, we have $0 \leq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq B^\star$ for $t \in \{t_1, t_2\}$. The claim thus holds.

Now assume the claim holds for round $j-1$. W.l.o.g. assume $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1)$. Let p be a project approved by 2 such that $c(p) \leq b_j$. Then, we set $\pi_j = \{p\}$. We can distinguish two cases: First assume 1 also approves p. Then, the claim follows directly from the induction hypothesis. Now, assume 1 does not approve p. Then

$$\begin{aligned} sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star &= sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star \\ &\leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) < sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2). \end{aligned}$$

Furthermore, we assumed

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \text{ and } c(p) \leq B^\star,$$

hence

$$\begin{aligned} sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) &= sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + c(p) \\ &\leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star = sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star. \end{aligned}$$

Therefore, the claim holds.

Now, we know $sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \geq \sum_{j'=1}^{j} c(\pi_{j'})$. Together with the claim, this implies that $\lim_{j\to+\infty}(sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)) = +\infty$ as well as $\lim_{j\to+\infty}(sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)) = +\infty$. We observe that

$$\lim_{j\to+\infty}\left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)}\right) = \lim_{j\to+\infty}\left(\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)}\right) = 1.$$

Therefore, the proposition follows from the following inequality:

$$\frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \leq \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2)}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)} \leq \frac{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star}{sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1)}.$$

□

Proof of Proposition 5

Proof. Assume w.l.o.g. that the agents are called 1,2 and 3. We have to distinguish two cases: either there are two or three types. Assume first that there are only two types t_1 and t_2. We can assume that agent 1 has type t_1 and agent 2 and 3 have type t_2. We claim that there exists a solution $\boldsymbol{\pi}$ such that for every round j, we can guarantee:

$$sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - B^\star \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star. \quad (7)$$

Convergence to equal-satisfaction follows from this equation analogues to Proposition 4.

Let us prove the claim by induction. For the first round, it is clear that whichever non-empty budget allocation has been chosen, we have $0 \leq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq B^\star$ for $t \in \{t_1, t_2\}$. The claim thus holds.

Now assume the claim holds for round $j-1$. Assume first that $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2)$. Then, we can just set $\pi_j = A_j(1)$. This guarantees

$$sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq B^\star.$$

Together with the induction hypothesis, this implies (7).

Now, assume $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1)$ If $A_j(1) = A_j(2) = A_j(3)$, then the difference in satisfaction will not change independently of the winning allocation, hence (7) follows from the induction hypothesis. Now assume it is not that case that $A_j(1) = A_j(2) = A_j(3)$ holds. As all ballots are exhaustive we cannot have $A_j(i) \subsetneq A_j(i^*)$ for $i, i^* \in \{1, 2, 3\}$, therefore there must be a project p that is not approved by all three agents. We claim that, moreover, there is a project p^* in $A_j(2) \cup A_j(3)$ that is not approved by 1. Assume otherwise. Then p must be approved by 1 but not approved by either 2 or 3. We assume w.l.o.g. that 2 does not approve p. This implies that $A_j(1) \cap A_j(2)$ is not exhaustive. Therefore $A_j(1) \cap A_j(2) \subsetneq A_j(2)$. In other words, there is a project p^* as desired that is approved by 2 but not by 1. Now let $\boldsymbol{\pi} = \{p^*\}$. By definition, we have

$$0 = sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq sat_j^m j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq B^\star.$$

Together with the induction hypothesis, this implies (7). This concludes the case that there are only two types.

Now, assume that there are three types t_1, t_2 and t_3. Further we assume w.l.o.g. that agent i has type t_i for all $i \in \{1, 2, 3\}$.

We claim that there exists a solution $\boldsymbol{\pi}$ such that for every round j, we can guarantee for any $i, i^* \in \{1, 2, 3\}$:

$$|sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_i) - sat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_{i^*})| \leq 2B^\star. \quad (8)$$

It is straightforward to check that convergence to equal-satisfaction follows from this equation analogues to Proposition 4.

Let us prove the claim by induction. For the first round, it is clear that whichever non-empty budget allocation has been chosen, we have $0 \leq sat_1(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq B^\star$ for $t \in \{t_1, t_2, t_3\}$. The claim thus holds.

Now assume the claim holds for round $j-1$. Further assume w.l.o.g. that

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3). \quad (9)$$

We observe that the induction hypothesis implies

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + 2B^\star.$$

Therefore, we must have either

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + B^\star \quad (10)$$

or

$$sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) + B^\star. \quad (11)$$

Assume first that (11) holds. Then, we set $\pi_j = A_j(1)$. From this we get

$$sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j^m j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq B^\star,$$
$$sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_j^m j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq B^\star.$$

Together with (9) this implies (8) for $i = 1$ and $i^* = 2, 3$. It remains to show that it also holds for $i = 2$ and $i^* = 3$. Assume $sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_3)$, then (8) follows from $sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_3)$ and $sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq B^\star$. So, assume $sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2)$. Then (8) follows from (11) and $sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq B^\star$.

Now assume that (10) holds. By the same argument as above we can find a p^* that is in $A_j(2) \cup A_j(3)$ but not in $A_j(1)$. Let $\pi_j = \{p^*\}$. Assume first that $p^* \in A_j(1)$. Then,

$$sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_3) \leq sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_1) \leq B^\star$$

together with (9) implies (8) for all i and i^*. So assume $p^* \in A_j(2) \setminus A_j(1)$. Then $sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_1) = sat_j^m(\boldsymbol{I}, \boldsymbol{\pi}, t_3) = 0$ and $sat_j^m j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq B^\star$. This implies, together with the induction hypothesis, (8) for $i = 1$ and $i^* = 3$ and, together with (9), (8) for $i = 2$ and $i^* = 3$. Finally, $sat_j^m j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq B^\star$ together with (10) implies (8) for $i = 1$ and $i^* = 2$.

□

Proof of Lemma 6

Proof. Consider a round $j \in \{1, \dots, k\}$ corresponding to the budgeting problem $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$. We show that there exists $\pi \in \mathcal{A}(I_j)$ such that $0 < rsat_j^m(\boldsymbol{I}, \pi, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi, t_2)$. The other direction follows by symmetry.

Given that I_j has knapsack ballots, for every agent $i \in \mathcal{N}$, we have $A_j(i) \in \mathcal{A}(I_j)$. For any budget allocation $\pi \in \mathcal{A}(I_j)$ and type $t \in \mathcal{T}$, let $s_\pi^t = |\{i \in t \mid A_j(i) = \pi\}|$ be the number of agents of type t that approve exactly the budget allocation π.

Let π^* be any budget allocation such that $s_{\pi^*}^{t_1} \neq 0$. Now, if $rsat_j^m(\boldsymbol{I}, \pi^*, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi^*, t_2)$ holds, the lemma holds as well. Therefore, assume

$$rsat_j^m(\boldsymbol{I}, \pi^*, t_1) < rsat_j^m(\boldsymbol{I}, \pi^*, t_2). \tag{12}$$

Now, observe that by definition we have

$$rsat_j^m(\boldsymbol{I}, \pi^*, t) = \frac{1}{|t|} \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^t \alpha(\pi, \pi^*) \tag{13}$$

where $\alpha(\pi, \pi^*)$ is the relative overlap of π and π^*, defined by

$$\alpha(\pi, \pi^*) = \frac{c(\pi \cap \pi^*)}{c(\pi)}.$$

We observe that $0 \leq \alpha(\pi, \pi^*) \leq 1$ holds for all π by definition. Now, from (12) and (13) we have:

$$\frac{1}{|t_1|} \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_1} \alpha(\pi, \pi^*) < \frac{1}{|t_2|} \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_2} \alpha(\pi, \pi^*)$$

$$\Leftrightarrow \quad \frac{\sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_1} \alpha(\pi, \pi^*)}{\sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_2} \alpha(\pi, \pi^*)} < \frac{|t_1|}{|t_2|}. \tag{14}$$

Now, let $A^1 := \{\pi \mid \alpha(\pi, \pi^*) = 1\}$ be the set of budget allocations π for which $\alpha(\pi, \pi^*) = 1$. Then, we can write (14) as

$$\frac{\sum_{\pi \in A^1} s_\pi^{t_1} + \sum_{\pi \notin A^1} s_\pi^{t_1} \alpha(\pi, \pi^*)}{\sum_{\pi \in A^1} s_\pi^{t_2} + \sum_{\pi \notin A^1} s_\pi^{t_2} \alpha(\pi, \pi^*)} < \frac{|t_1|}{|t_2|}.$$

Now, for this to hold, we must either have

$$\frac{\sum_{\pi \in A^1} s_\pi^{t_1}}{\sum_{\pi \in A^1} s_\pi^{t_2}} < \frac{|t_1|}{|t_2|}$$

or there must be sufficiently many budget allocations $\pi \notin A^1$ with $\frac{s_\pi^{t_1}}{s_\pi^{t_2}} < \frac{|t_1|}{|t_2|}$ to make the inequality true. As we know that $\alpha(\pi, \pi^*) < 1$, dropping $\alpha(\pi, \pi^*)$ only increases the influence of these projects. Therefore, in both cases, we can find a set of budget allocations $A^1 \subseteq A \subseteq \mathcal{A}(I_j)$ such that the following holds:

$$\frac{\sum_{\pi \in A} s_\pi^{t_1}}{\sum_{\pi \in A} s_\pi^{t_2}} < \frac{|t_1|}{|t_2|}. \tag{15}$$

On the other hand, since for any type t, we have $|t| = \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^t$, (15) can be rewritten

$$\frac{\sum_{\pi \in A} s_\pi^{t_1}}{\sum_{\pi \in A} s_\pi^{t_2}} < \frac{\sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_1}}{\sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_2}}.$$

As $\sum_{\pi \in A} s_\pi^{t_2} \leq \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_2}$ holds by definition, we must have $\sum_{\pi \in A} s_\pi^{t_1} < \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_1}$. Thus, there must exists a $\pi_0 \in \mathcal{A}(I_j) \setminus A$ such that $s_{\pi_0}^{t_1} \neq 0$. Since $\pi_0 \notin A$, we have $\alpha(\pi_0, \pi^*) < 1$, which implies $\pi_0 \neq \pi^*$.

Consider then $\pi^{*\prime} = \pi^* \setminus \pi_0$. Since $\alpha(\pi_0, \pi^*) < 1$, $\pi^{*\prime} \neq \emptyset$ and $rsat^m(\boldsymbol{I}, \pi^{*\prime}, i) > 0$ for at least one agent i of type t_1 (one such that $A_j(i) = \pi_0$ for instance, which must exist as $s_{\pi_0}^{t_1} \neq 0$). Therefore, if

$$rsat_j^m(\boldsymbol{I}, \pi^{*\prime}, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi^{*\prime}, t_2)$$

holds, then the lemma holds. Hence, we assume

$$rsat_j^m(\boldsymbol{I}, \pi^{*\prime}, t_1) < rsat_j^m(\boldsymbol{I}, \pi^{*\prime}, t_2).$$

Then, we have

$$rsat_j^m(\boldsymbol{I}, \pi^*, t_1) + rsat_j^m(\boldsymbol{I}, \pi^{*\prime}, t_1) < rsat_j^m(\boldsymbol{I}, \pi^*, t_2) + rsat_j^m(\boldsymbol{I}, \pi^{*\prime}, t_2). \tag{16}$$

It follows from (13) and (16) as before that

$$\frac{\sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_1} (\alpha(\pi, \pi^*) + \alpha(\pi, \pi^{*\prime}))}{\sum_{\pi \in \mathcal{A}(I_j)} s_\pi^{t_2} (\alpha(\pi, \pi^*) + \alpha(\pi, \pi^{*\prime}))} < \frac{|t_1|}{|t_2|}.$$

Now, because π^* and $\pi^{*\prime}$ are disjoint, we know all factors $\alpha(\pi, \pi^*) + \alpha(\pi, \pi^{*\prime})$ are smaller or equal 1. Hence we can conclude as above that there must be a set of budget allocations $A' \subseteq \mathcal{A}(I_j)$ such that $\alpha(\pi, \pi^*) + \alpha(\pi, \pi^{*\prime}) = 1$ implies $\pi \in A'$ and the following holds

$$\frac{\sum_{\pi \in A'} s_\pi^{t_1}}{\sum_{\pi \in A'} s_\pi^{t_2}} < \frac{|t_1|}{|t_2|}. \tag{17}$$

It follows again from $|t| = \sum_{\pi \in \mathcal{A}(I_j)} s_\pi^t$ and (17) that there must be another budget allocation π_1 such that $s_{\pi_1}^{t_1} > 0$ and $\alpha(\pi_1, \pi^*) + \alpha(\pi_1, \pi^{*\prime}) < 1$, which implies $\pi_1 \notin \{\pi^*, \pi^{*\prime}\}$. Then, as before, we have $rsat_j^m(\boldsymbol{I}, \pi_1 \setminus (\pi^* \cup \pi^{*\prime}), i) > 0$ for some agent i of type t_1 and hence if

$$rsat_j^m(\boldsymbol{I}, \pi_1 \setminus (\pi^* \cup \pi^{*\prime}), t_1) \geq rsat_j^m(\boldsymbol{I}, \pi_1 \setminus (\pi^* \cup \pi^{*\prime}), t_2)$$

then the lemma holds. Otherwise, we can iterate the construction.

As there are only finitely many allocations, this construction must lead, after finitely many steps, to an allocation $\pi^\star$ such that $0 < rsat_j^m(\boldsymbol{I}, \pi^\star, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi^\star, t_2)$. ☐

Proof of Theorem 7

Proof. Let us call the two types t_1 and t_2. We claim that there exists a solution $\boldsymbol{\pi}$ such that for every round j, we can guarantee:

$$rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - 1 \leq rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + 1.$$

We will prove the claim by induction. For the first round, we clearly have $0 \leq rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t) \leq 1$ for $t \in \{t_1, t_2\}$, hence the claim holds. Now assume the claim holds for round $j - 1$. W.l.o.g. assume

$$rsat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq rsat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t_1). \tag{18}$$

By Lemma 6, we know that there is a feasible budget allocation π for round j such that $rsat_j^m(\boldsymbol{I}, \pi, t_1) \leq rsat_j^m(\boldsymbol{I}, \pi, t_2)$. Assume that $\pi_j = \pi$. Then, for both types $t \in \{t_1, t_2\}$, we have

$$rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t) = rsat_{j-1}(\boldsymbol{I}, \boldsymbol{\pi}, t) + rsat_j^m(\boldsymbol{I}, \pi_j, t)$$

Since $rsat_j^m(\boldsymbol{I}, \pi_j, t_1) \leq rsat_j^m(\boldsymbol{I}, \pi_j, t_2)$, and by the induction hypothesis, it follows that

$$rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) - 1 \leq rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2).$$

On the other hand, from (18) and the fact that $rsat_j^m(\boldsymbol{I}, \pi_j, t_2) \leq 1$ holds by definition, we obtain

$$rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_2) \leq rsat_j(\boldsymbol{I}, \boldsymbol{\pi}, t_1) + 1.$$

Now, in each round, we know from Lemma 6 that we can always select a budget allocation that improves the relative satisfaction of type t (t being the type with the lowest relative satisfaction) by at least $1/(B^\star|t|)$. Hence, the relative satisfaction of both types goes to infinity, while the difference is always less than 1. This concludes the proof as it implies that $\boldsymbol{\pi}$ converges to equal relative satisfaction. □

Proof of Corollary 8

Proof. The idea of the proof is, that we can find two budgeting allocations π_1 and π_2 as described in Lemma 6 and such that there are two agents $i_1, i_2 \in \mathcal{N}$ with $A_j(i_1) \setminus \pi_1 = \emptyset$ and $A_j(i_2) \setminus \pi_2 = \emptyset$ by applying Lemma 6 several times. Let $I_j = \langle \mathcal{P}_j, b_j, A_j \rangle$ be the budgeting problem in round j. We claim that, under the given assumptions, in every round j there exist two feasible budget allocations $\pi_1, \pi_2 \in \mathcal{A}(I_j)$ such that

$$0 < rsat_j^m(\boldsymbol{I}, \pi_1, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi_1, t_2),$$
$$rsat_j^m(\boldsymbol{I}, \pi_2, t_1) \leq rsat_j^m(\boldsymbol{I}, \pi_2, t_2) > 0,$$

and for which there are two agents $i_1, i_2 \in \mathcal{N}$ with $A_j(i_1) \setminus \pi_1 = \emptyset$ and $A_j(i_2) \setminus \pi_2 = \emptyset$. We will only prove the existence of π_1, that of π_2 follows by symmetry. Indeed, from Lemma 6, we know that there exists $\pi_1^1 \in \mathcal{A}(I_j)$, such that

$$0 < rsat_j^m(\boldsymbol{I}, \pi_1^1, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi_1^1, t_2).$$

If there is no agent $i \in \mathcal{N}$ such that $A_j(i) \setminus \pi_1^1 = \emptyset$, we can consider the budgeting problem $I_j^1 = \langle \mathcal{P}_j^1, b_j, A_j^1 \rangle$ where $\mathcal{P}_j^1 = \mathcal{P}_j \setminus \pi_1^1$ and for all agents $i \in \mathcal{N}$ we have $A_j^1(i) = A_j(i) \setminus \pi_1^1$. By assumption, $A_j^1(i) \neq \emptyset$ for all agents $i \in \mathcal{N}$. Therefore, I_j^1 is a budgeting problem with two types and non-empty knapsack ballots and we can apply Lemma 6 once again, to obtain a budget allocation $\pi_1^2 \in \mathcal{A}(I_j^1)$. such that

$$0 < rsat_j^m(\boldsymbol{I}, \pi_1^2, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi_1^2, t_2).$$

By definition, we know that $\pi_1^1 \cap \pi_1^2 = \emptyset$, therefore we have

$$0 < rsat_j^m(\boldsymbol{I}, \pi_1^1 \cup \pi_1^2, t_1) = rsat_j^m(\boldsymbol{I}, \pi_1^1, t_1) + rsat_j^m(\boldsymbol{I}, \pi_1^2, t_1) \geq rsat_j^m(\boldsymbol{I}, \pi_1^1, t_2) + rsat_j^m(\boldsymbol{I}, \pi_1^2, t_2) = rsat_j^m(\boldsymbol{I}, \pi_1^1 \cup \pi_1^2, t_2).$$

Now, if there is no agent i such that $A_j(i) \setminus \pi_1^1 \cup \pi_1^2 = \emptyset$, we can apply Lemma 6 again until we have $A_j(i) \setminus (\pi_1^1 \cup \pi_1^2 \cup \ldots) = \emptyset$ for some agent.

Finally, note that if all ballots are exhaustive, then $A_j(i) \setminus \pi = \emptyset$ can only hold for an exhaustive budget allocation π. □